Study of multiple target defense differential games using mode switching strategies

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***Abstract*—In this paper, we study a variation of the Active Target-Attacker-Defender (ATAD) differential game involving multiple targets, an attacker and a defender. Our model allows for (i) a capability of the defender to switch roles from *rescuer* (rendezvous with all the targets) to *interceptor* (intercepts the attacker) and vice-versa; (ii) the attacker to continuously pursue the closest target (which can change during the course of the game). We assume that the mode of the defender (rescue or interception) defines the mode of the game itself. Using the framework of Games of a Degree, we first analyze the game within each mode. More specifically, the objectives of the players are taken as a combination of weighted Euclidean distances and penalties on their control efforts. We model the interaction of the players within each mode as a linear quadratic differential game (LQDG) and obtain the open-loop Nash equilibrium strategies. We then use the receding horizon approach to enable switching between the modes to obtain switching strategies for the players. By partitioning the matrices associated with the Riccati differential equations we obtain geometric characterization of the trajectories of the players. In particular, under mild restrictions on the problem parameters we show that (i) in the interception mode, the attacker and its closest target move in a straight line; (ii) in the interception mode, the closest target and the remaining targets undergo parallel evolution; (iii) in the rescue mode, the distance between the closest target and other targets, and their orientation remains constant; (iv) there exists a bound on the length of the planning horizon which ensures that the attacker locks on to a target in the interception mode. We illustrate our results with numerical simulations. Experimental results involving multiple autonomous differential drive mobile robots are presented.**

***Index Terms*—Pursuit evasion differential games, Target-Attacker-Defender differential games, Switching strategies, Nash equilibrium, Receding horizon approach, Autonomous multi-agent systems.**

I. Introduction

The study of autonomous multi-agent interactions has received considerable interest in the recent years. This is mainly due to their applicability in modeling complex strategic phenomena arising in areas such as surveillance, rescue missions, combat operations, navigation, and analysis of biological behaviors. This paper is concerned with analyzing strategic situations observed in the engineering applications such as, a defense system protecting critical infrastructures (e.g., air crafts, naval ships) against attacks from incoming missiles, interceptor defending an asset against intrusions, and biological behavior such as mothers protecting young from potential attacks by the predators.

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A common feature in the above strategic situations is the presence of multiple agents which are at conflict or cooperation that evolves over time. These situations are usually analyzed using the mathematical framework of pursuit-evasion games with three players–Target, Attacker and Defender–and referred to as a Target-Attacker-Defender (TAD) game. Here, the goal of the attacker is to *capture* the target which tries to evade the attacker, and the goal of the defender is to *intercept* the attacker before the attacker captures the target. In a TAD game the target is assumed to be non-reactive (stationary or moves on a prescribed trajectory), and when the target is maneuverable then the interaction is referred to as an Active Target-Attacker-Defender (ATAD) game; see references in subsection I-B. Rescue type of interactions in the scenarios mentioned above can be modeled using the framework of a Prey-Protector-Predator (P3) game which was introduced in [1]. Here, the goal of the protector is to rendezvous with the prey in order to *rescue* the prey before it is captured by the predator. In a P3 interaction, when the players prey, protector and predators are seen as target, defender and attacker respectively, the role of the defender is to rescue the target instead of intercepting the attacker.

A. Contributions

A majority of the existing literature on (A)TAD and P3 games consider 3-player engagements. In the real world applications, for example, combat operations, rescue missions, and protection of young and coordinated hunting in the animal world often involve multiple ($n \geq 3$) players. Further, in almost all the existing works, the interactions between the players are fixed throughout the duration of the game. In the real-world scenarios, these interactions often change during the course of the engagement. For example, the defender may find it economical to rescue the targets instead of intercepting the attacker from the onset of the game. Only at an opportune moment, when the threat level reaches a certain threshold, the defender may want to switch to intercepting the attacker. Further, due to the presence of multiple targets, the attacker may want to dynamically update the target it wants to capture as the game evolves in time.

The contribution of this paper is to introduce a framework for studying dynamically evolving multi-agent interactions of ATAD type. In particular, the novelty of our work lies in the consideration of the presence of multiple targets, and a flexible (and powerful) capability of the defender to autonomously switch roles from being a rescuer to interceptor and vice-versa. Introducing these two features naturally leads to challenging questions such as (i) can the trajectories of the attacker and the target it pursues be geometrically characterized? (ii) how do the trajectories of targets evolve when the defender acts as a rescuer and as an interceptor? (iii) under what conditions will the attacker locks on to a target and pursues it forever? To address these questions we consider a model where players engage in two types of interactions, also called as *modes*, based on the role of the defender. In the *rescue* mode, the defender attempts rendezvous with the targets, whereas in the *interception* mode it tries to intercept the attacker. The attacker tries to capture a closest target and all targets try to evade the attacker. The defender is capable of autonomously switching the roles based on the state of the game. Our work distinguishes from the existing literature where the interactions are fixed for the entire duration of the engagement. To achieve our objective, first we fix the interactions of the players in one of the modes alone. Using the Games of a Degree approach, the interaction among the players in a mode is formulated as a finite horizon non-zero sum linear quadratic differential game (LQDG); similar approach was followed in the works [2] and [3], and the open-loop Nash equilibrium control strategies of the players are computed. To facilitate switching between the modes, we adopt the receding horizon approach to obtain switching strategies of the players. The main results of our paper are summarized as follows.

1) In the interception mode, we show in Theorem 3 that the attacker and its closest target move on a straight line joining their initial locations.
2) In the interception mode, we show in Theorem 4 when the targets are identical, then the closest target (to the attacker) and other targets undergo parallel evolution.
3) In the interception mode, we show in Lemma 1 that the distance between the closest target (to the attacker) and other targets either increase or decrease depending upon the bounds placed on the planning horizon length. Using this result, and with a particular form of defender's switching function we show in Theorem 7 that the interception mode is invariant, and the attacker locks on to a target for the remaining duration of the game.
4) In the rescue mode, we show in Theorem 6 that the distance between the closest target (to the attacker) and the other targets remains constant.

The paper is organized as follows. In section II, we present dynamics of the players and their interactions. In section III, we solve the LQDG assuming that the mode of the game is restricted to either rescue or interception alone and derive the open-loop Nash equilibrium strategies of the players. In section IV, we augment the open-loop Nash equilibrium strategies with receding horizon approach to enable switching and provide an algorithm for computing the switching strategies of players. In section V we analyze the switching strategies and provide results related to the behavior of the players. In section VI, we illustrate our results with numerical simulations. Towards a practical realization of our study, in section VII, we illustrate our results taking differential drive mobile robots (DDMR) as players. Section VIII provides concluding remarks and a summary of future research.

B. An overview of the related literature

TAD type interactions were studied in [4], [5] in the context of defending ships from an incoming torpedo using counter-weapons. In [6], a two-player differential game of target defence is studied, where the objective of one player is to drive the state of the system to reach the target whereas the other player requires the state to avoid the target. A TAD type interaction referred to as the lady, the bandits and the body-guards was proposed in [7]. In [8], the authors study an ATAD terminal game and propose attacker strategies for evading the defender while continuing to pursue the target. In [2], the authors study the problem of defending an asset. Here, the interactions are modeled as a LQDG. The authors propose moving horizon strategies for different configurations of the target, namely when it is stationary, moves in prescribed trajectory and maneuverable. In [9], a guidance law for defending a non-maneuverable aircraft is proposed. An algorithm for the real-time target guarding problem was studied in [10]. In [11], [12], line-of-sight and other guidance laws are presented for defending aerial targets. In [13] [14], [15], [16] the authors consider various cooperation scenarios between the aircraft (target) and the defensive missile (defender) against the incoming homing missile (attacker). More specifically, it was shown [13] that the target can lure the attacker to get intercepted by the defender even though maneuverability of the defender is limited. P3 type of interactions were investigated by Oyler et al. [1]. The authors solve the game by characterizing the dominance regions using Apollonius circles in the presence of obstacles. In [17], dominance regions have also been used to study the effect of a vision-guided predator that acts to prevent rendezvous of multiple protector and prey robots.

In a series of works [18], [19], [20], [21], the authors consider a ATAD game where a homing missile (attacker) tries to pursue an aircraft (target), and a defender missile aims at intercepting the attacker in order to protect the target. In particular, they study cooperative mechanisms between the target-defender team against the attacker so that the defender can intercept the attacker before the attacker can capture the target. In [22], the ATAD interaction is posed as a zero-sum differential game between the defender-target team and the attacker. A complete characterization for the target's escape set and synthesis of close-loop state feedback strategies for the players are provided. In [23] the same game is studied and the authors construct barrier surfaces and characterize the escape and capture regions for the target. In [24], the authors study a TAD interaction as a LQDG and provides closed-form solutions for players' strategies through the analysis of the associated coupled Riccati differential equations.

Related to the literature on role switching in ATAD games, in the recent work [25] the authors study a 3-player ATAD game where the survivability of the attacker is of importance. In this model, the attacker is allowed to switch from pursuing the target to evading the defender at an opportune moment. In [26] the defender's strategies force the attacker to retreat instead of engaging the target . In [27], the authors study the possibility of role switch as well as the cooperation between the target and defender. In almost all the above works related to (A)TAD game, the role of the defender is to intercept the attacker.

Preliminary conference version of this paper appeared in [28] where a 3-player game is studied and does not consider the presence of multiple targets. This paper goes much beyond the work [28], both in content and scope, by providing proofs for the analytical characterizations of trajectories, and illustrations with experiments.

C. Notation

Throughout this paper, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. The symbol $\otimes$ denotes the Kronecker product. The transpose of a vector or matrix E is denoted by E'. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $||x||_2 = \sqrt{x'x}$. For any $x \in \mathbb{R}^n$ and $S \in \mathbb{R}^{n\times n}$, we denote the quadratic term $x'Sx$ by $||x||^2_S$.

II. Multiple active target attacker defender differential game

In this section, we describe the interactions and dynamics of the players, and provide the dynamic game methodology for modeling players' interactions. We use the terms player/agent interchangeably throughout the paper.

A. Dynamics of players

We consider a team of n active targets which are pursued by one attacker. We assume the availability of one defender whose task is to either save or *rescue* all the targets or to *intercept* the attacker. We denote the set of n target vehicles by $\mathscr{A} := \{a_1, a_2, \cdots, a_n\}$, the defender by b and the attacker by c. The set of players is denoted by $\mathscr{P} := \mathscr{A} \cup \{b, c\}$. We assume that the players interact in a two-dimensional plane. The dynamics of each player is governed by the following single integrator dynamics:

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{y}_i(t) \end{bmatrix} = \begin{bmatrix} u_{ix}(t) \\ u_{iy}(t) \end{bmatrix}, \quad \begin{bmatrix} x_i(0) \\ y_i(0) \end{bmatrix} = \begin{bmatrix} x_{i0} \\ y_{i0} \end{bmatrix} \tag{1}$$

where $(x_i(t), y_i(t)) \in \mathbb{R}^2$ is the position vector of the player $i \in \mathscr{P}$ at time t, $(u_{ix}(t), u_{iy}(t)) \in \mathbb{R}^2$ represents the control input of player i at time t, and $(x_{i0}, y_{i0}) \in \mathbb{R}^2$ represents the initial position vector of player i. We denote the state and control vector of player $i \in \mathscr{P}$ as:

$$X_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}, \quad u_i(t) = \begin{bmatrix} u_{1i}(t) \\ u_{2i}(t) \end{bmatrix}. \tag{2}$$

By denoting $X(t) = [X'_{a_1}(t) \;\; X'_{a_2}(t) \;\; \cdots \;\; X'_{a_n}(t) \;\; X'_b(t) \;\; X'_c(t)]'$, the dynamic interaction environment of the players can be written compactly as follows:

$$\dot{X}(t) = \Big(\sum_{j=1}^{n} B_{a_j} u_{a_j}(t)\Big) + B_b u_b(t) + B_c u_c(t), \tag{3}$$

where $B_{a_j} = \begin{bmatrix} d_1 & d_2 & \cdots & d_n & 0 & 0 \end{bmatrix}' \otimes I \in \mathbb{R}^{2N\times 2}$ with $d_j = 1$, $d_l = 0$, $\forall\, l \neq j$, $B_b = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}' \otimes I \in \mathbb{R}^{2N\times 2}$, $B_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}' \otimes I \in \mathbb{R}^{2N\times 2}$, where $N = n+2$ and I represents 2×2 identity matrix.

Remark 1. The methodology and results presented in the paper can be extended easily to an n-dimensional setting.

B. Players' interactions as a differential game

In our paper the interactions between the players are described as follows:

I1. The attacker always tries to capture a target which is at the closest distance to it.

I2. The defender can operate in two modes namely rescue or interception modes. In the interception mode, the defender tries to intercept the attacker, whereas in the rescue mode the defender ties to rendezvous with all the targets (in order to save them). The defender is capable of switching the operational modes autonomously depending upon the state of the game.

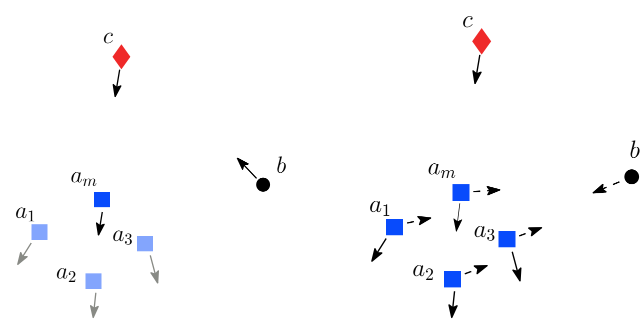
I3. The targets always try to evade the attacker in the interception mode. In the rescue mode, besides evading the attacker they also attempt to rendezvous with the defender.

Figure 1 illustrates the interaction among the players in both the operational models. Our work distinguishes from the existing literature due to presence of the following three features in the interactions (I1-I3).

F1. The attacker always pursues a target which is at minimum distance to it. This implies that the target with whom the attacker is in direct conflict keeps changing with time.

F2. As the defender can autonomously switch operating in rescue mode to interception mode and vice-versa, the players with whom the defender is in direct conflict/cooperation also changes with time.

F3. The targets are in conflict with the attacker both in rescue and interception modes, and they are in cooperation with the defender in rescue mode.



(a) Interception mode (b) Rescue mode

Fig. 1: Interaction of players in both the modes. a_m is a target which is at a minimum distance to the attacker c.

In the ATAD differential games literature, usually two approaches are followed for analyzing the interactions of the type (I1-I3). The *Game of a Kind* answers questions related to the eventual outcome of the game such as capture of a target by the attacker, interception of the attacker by the defender, and if the targets can be rescued by the defender. In other words, in this approach there are usually finitely many outcomes of the game, and the discrete nature of the payoffs translate to imposing certain hard constraints on the strategies of the players; see [2, Section II]. As a result, the solution methods lead to non-differentiable value functions which may not be compatible with the information structures [29], e.g., open-loop and feedback, which are typically used in the study of differentiable games. On the other hand, in the *Game of a Degree* approach, performance metrics–for instance, Euclidean distance between the players–are used towards quantifying the result of the game. Introduction of the performance metrics have a softening effect on the constraints imposed on the decision variables of the players; see [2, Section II]. This approach leads to efficient solution methods for analyzing real world problems that are modeled as ATAD type differential games. For the reasons mentioned above, we use the Games of a Degree methodology for analyzing the interactions of the type (I1-I3). In the next section, we first study the game where the interactions of the players are fixed in the rescue mode or interaction mode alone.

III. Analysis in rescue or interception mode

Let the target which is at a minimum distance to the attacker at time $t=0$ be denoted by a_m. Then a_m satisfies

$$a_m := \arg\min_{a\in\mathscr{A}} ||X_c(0)-X_a(0)||_2. \tag{4}$$

Firstly, we consider the setting where the interaction network of the players is fixed to be either in the rescue or interception mode for a time duration $T>0$. We assume that the attacker pursues the target a_m throughout the time duration $[0,T]$. Further, we also assume that the defender operates in one of these modes for the duration $[0,T]$. In the rescue mode, the defender minimizes the sum of weighted Euclidean distances to all the targets. In this mode, the targets in $\mathscr{A}$ minimize and maximize their weighted Euclidean distance with the defender and the attacker respectively. In the interception mode, the defender minimizes its weighted Euclidean distance with the attacker while the targets in $\mathscr{A}$ maximize their weighted Euclidean distances with the attacker. All the players simultaneously minimize the energy expenditure i.e., the control effort to be consumed in (both) the modes. The performance metric to be minimized by player $i\in\mathscr{P}$ is then given by

$$J_i(u_{a_1}(.),\cdots,u_{a_n}(.),u_b(.),u_c(.)) = G_i(T)+\int_0^T L_i(t)dt, \tag{5}$$

where the terminal cost $G_i(T)$ and running cost $L_i(t)$ of player $i\in\mathscr{P}$ are defined as follows:

for $j\in\{1,2,\cdots,n\}$

$$\begin{aligned}G_{a_j}(T) &= \frac{\alpha_R}{2}||X_{a_j}(T)-X_b(T)||^2_{Q_{a_jbT}}\\ &\quad -\frac{1}{2}||X_{a_j}(T)-X_c(T)||^2_{Q_{a_jcT}} = \frac{1}{2}||X(T)||^2_{\widetilde{Q}_{a_jT}},\end{aligned} \tag{6}$$

$$\begin{aligned}L_{a_j}(t) &= \frac{1}{2}||u_{a_j}(t)||^2_{R_{a_j}}+\frac{\alpha_R}{2}||X_{a_j}(t)-X_b(t)||^2_{Q_{a_jb}}\\ &\quad -\frac{1}{2}||X_{a_j}(t)-X_c(t)||^2_{Q_{a_jc}}\\ &= \frac{1}{2}||u_{a_j}(t)||^2_{R_{a_j}}+\frac{1}{2}||X(t)||^2_{\widetilde{Q}_{a_j}},\end{aligned} \tag{7}$$

$$\begin{aligned}G_b(T) &= \frac{\alpha_R}{2}\sum_{j=1}^n ||X_{a_j}(T)-X_b(T)||^2_{Q_{ba_jT}}\\ &\quad + \frac{\alpha_I}{2}||X_b(T)-X_c(T)||^2_{Q_{bcT}} = \frac{1}{2}||X(T)||^2_{\widetilde{Q}_{bT}},\end{aligned} \tag{8}$$

$$\begin{aligned}L_b(t) &= \frac{1}{2}||u_b(t)||^2_{R_b}+\frac{\alpha_R}{2}\sum_{j=1}^n ||X_{a_j}(t)-X_b(t)||^2_{Q_{ba_j}}\\ &\quad +\frac{\alpha_I}{2}||X_b(t)-X_c(t)||^2_{Q_{bc}}\\ &= \frac{1}{2}||u_b(t)||^2_{R_b}+\frac{1}{2}||X(t)||^2_{\widetilde{Q}_b},\end{aligned} \tag{9}$$

$$G_c(T) = \frac{1}{2}||X_{a_m}(T)-X_c(T)||^2_{Q_{ca_mT}} = \frac{1}{2}||X(T)||^2_{\widetilde{Q}_{cT}}, \tag{10}$$

$$\begin{aligned}L_c(t) &= \frac{1}{2}||u_c(t)||^2_{R_c}+\frac{1}{2}||X_{a_m}(t)-X_c(t)||^2_{Q_{ca_m}}\\ &= \frac{1}{2}||u_c(t)||^2_{R_c}+\frac{1}{2}||X(t)||^2_{\widetilde{Q}_c}.\end{aligned} \tag{11}$$

Here, the matrices Q_{ijT} and Q_{ij}, $i,j\in\mathscr{P}$, $i\neq j$ are symmetric 2×2 matrices. Further, $\widetilde{Q}_{iT}$ and $\widetilde{Q}_i$ are $2N\times 2N$ symmetric matrices, and R_i are 2×2 symmetric and positive definite matrices for all $i\in\mathscr{P}$. The matrices $\widetilde{Q}_i$ and $\widetilde{Q}_{iT}$ have the similar structures except an additional subscript T, and are described as follows. $\widetilde{Q}_{a_j} = \left[\begin{array}{c|c} Q_1 & Q_2' \\ \hline Q_2 & Q_3\end{array}\right]$, where $Q_1 = \text{diag}\{q_{a_1},\cdots,q_{a_n}\}$ with $q_{a_i} = (\alpha_R Q_{a_jb}-Q_{a_jc})$ for $i=j$ and $q_{a_i}=0$ for $i\neq j$, $Q_2 = [\hat{q}_1,\cdots,\hat{q}_n]$ with $\hat{q}_j = \begin{bmatrix}-\alpha_R Q_{a_jb}\\ Q_{a_jc}\end{bmatrix}$, $\hat{q}_l = \begin{bmatrix}0\\0\end{bmatrix}$ $\forall$ $l\neq j$, and $Q_3 = \text{diag}\{\alpha_R Q_{a_jb}, -Q_{a_jc}\}$. $\widetilde{Q}_b = \left[\begin{array}{c|c} Q_4 & Q_5' \\ \hline Q_5 & Q_6\end{array}\right]$, where $Q_4 = \alpha_R$ $\text{diag}\{Q_{ba_1},\cdots,Q_{ba_n}\}$, $Q_5 = -\alpha_R\begin{bmatrix}Q_{ba_1} & Q_{ba_2} & \cdots & Q_{ba_n}\\ 0 & 0 & \cdots & 0\end{bmatrix}$, and $Q_6 = \begin{bmatrix}(\alpha_R\sum_{j=1}^n Q_{ba_j}+\alpha_I Q_{bc}) & -\alpha_I Q_{bc}\\ -\alpha_I Q_{bc} & \alpha_I Q_{bc}\end{bmatrix}$. $\widetilde{Q}_c = \left[\begin{array}{c|c} Q_7 & Q_8' \\ \hline Q_8 & Q_9\end{array}\right]$, where $Q_7 = \text{diag}\{q_{a_1},\cdots,q_{a_n}\}$ with $q_{a_i} = Q_{ca_m}$ for $a_i = a_m$ and $q_{a_i}=0$ for $a_i\neq a_m$; $Q_8 = [\hat{q}_{a_1},\cdots,\hat{q}_{a_n}]$ with $\hat{q}_{a_i} = \begin{bmatrix}0\\ -Q_{ca_m}\end{bmatrix}$ for $a_i = a_m$, $\hat{q}_{a_i} = \begin{bmatrix}0\\0\end{bmatrix}$ for $a_i\neq a_m$; and $Q_9 = \text{diag}\{0,Q_{ca_m}\}$. The dimension of the matrices Q_3, Q_6, Q_9 is 4×4; Q_1, Q_4, Q_7 is $2n\times 2n$ and of Q_2, Q_5, Q_8 is $4\times 2n$. The parameters $\alpha_R,\alpha_I\in\{0,1\}$, reflect the fact that when the game is in rescue mode the parameters are set to $(\alpha_R,\alpha_I)=(1,0)$, and they are set to $(\alpha_R,\alpha_I)=(0,1)$ in the interception mode.

We assume that all the players are aware of the objectives of themselves as well as the other players and have access to the state vector $x(t)$ for all $t\in[0,T]$. Each player $i\in\mathscr{P}$ solves the following optimal control problem

$$\min_{u_i(.)} J_i(u_{a_1}(.),\cdots,u_{a_n}(.),u_b(.),u_c(.)) \text{ subject to (3)}. \tag{12}$$

Due to linearity of the dynamics (3) and the quadratic nature of the performance metrics (5), problem (12) describes a $n+2$ finite horizon non-zero sum LQDG in rescue or interception mode. We seek to obtain Nash equilibrium strategies of the players.

Definition 1 (Nash equilibrium)**.** The strategy profile $(u^*_{a_1}(.),u^*_{a_2}(.),\cdots,u^*_{a_n}(.),u^*_b(.),u^*_c(.))$ is a Nash equilibrium for the linear quadratic differential game (12) if the following set of inequalities hold true for every $i\in\mathscr{P}$

$$J_i(u^*_i(.),u^*_{-i}(.))\leq J_i(u_i(.),u^*_{-i}(.)),\forall u_i(.). \tag{13}$$

Here, the notation $-i$ stands for players other than i, that is $-i := \mathscr{P}\backslash\{i\}$, and $u_{-i}(.)$ is the strategy profile of all the players in $\mathscr{P}$ excluding player i.

Remark 2. The (A)TAD game studied in [2] the players' interactions are modeled as a zero-sum linear quadratic differential game between the attacker and the defender-target team resulting in minmax strategies. In our paper, the defender's objectives are different in rescue and interception modes. Further, we do not assume cooperation between the targets and all the players individually minimize their objectives. So, Nash equilibrium is a natural choice for the outcome of the game described by (3) and (5).

It is well known in differential games literature [29] that the strategies of players depend upon the information available to the players while taking their decisions; referred to as information structure. Commonly, two types of information structures are used in differential games. In the open-loop information structure, the decisions of players are functions of time and the initial condition. In the feedback information structure, the decisions of players are functions of the state variable. Feedback strategies are adaptive and come with useful properties such as strong time consistency [29]. There exist methods for computing both the open-loop and feedback Nash equilibria [30]. However, in this paper due to the complexity of analysis, we restrict our attention to open-loop information structure. The open-loop Nash equilibrium strategies can be computed by jointly solving $N := n+2$ optimal control problems, given by (12) using the Pontryagin maximum principle. We thus have the following result from [30].

Theorem 1. *[30, Theorem 7.2] Consider the N player finite horizon LQDG described by (12) with open-loop information structure. Let there exist a solution set $\{P_i(t),\ i \in \mathscr{P}\}$ to the following N coupled Riccati differential equations*

$$\dot{P}_i(t) = P_i(t)\Big(\sum_{j=1}^{n}[S_{a_j}P_{a_j}(t)] + S_bP_b(t) + S_cP_c(t)\Big) - \widetilde{Q}_i, \quad (14)$$

where $P_i(T) = \widetilde{Q}_{iT}$ and $S_i = B_iR_i^{-1}B_i'$. The unique open-loop Nash equilibrium solution at time $t \in [0,T]$ for every initial state $X(0)$ is given by

$$u_i^*(t;X(0)) = -R_i^{-1}B_i'P_i(t)\Phi(t,0)X(0), \quad (15)$$
$$\dot{\Phi}(t,0) = \Big(-\sum_i S_iP_i(t)\Big)\Phi(t,0) = A_{cl}(t)\Phi(t,0),\ \Phi(0,0) = I.$$

Upon using the open-loop Nash equilibrium strategies, the closed loop system matrix is given by $A_{\mathrm{cl}}(t) = \big(-\sum_i S_iP_i(t)\big)$, and the closed-loop dynamic interaction environment (3) evolves according to

$$\dot{X}(t) = A_{\mathrm{cl}}(t)X(t),\ t \geq 0. \quad (16)$$

Remark 3. As we assumed that the matrix R_i is symmetric and positive definite, it can be easily verified that the cost function J_i given by (5) is strictly convex in $u_i(.)$ for all control functions $u_j(.)$ $j \neq i$ and for all the initial conditions X_0. This implies that the conditions obtained using the Pontryagin maximum principle are both necessary and sufficient.

Next, we discuss the conditions which guarantee the solvability of the coupled Riccati differential equations (14). Let us define

$$M = \begin{bmatrix} \mathbf{0} & -\mathbf{S} \\ -\mathbf{Q}' & \mathbf{0} \end{bmatrix},\quad \mathbf{S} = \begin{bmatrix} S_{a_1} & S_{a_2} & \cdots & S_{a_n} & S_b & S_c \end{bmatrix},$$
$$\mathbf{Q} = \begin{bmatrix} \widetilde{Q}'_{a_1} & \widetilde{Q}'_{a_2} & \cdots & \widetilde{Q}'_{a_n} & \widetilde{Q}'_b & \widetilde{Q}'_c \end{bmatrix},$$

$$H(T) = \begin{bmatrix} I_{2N\times 2N} & 0 & \cdots & 0 \end{bmatrix} e^{-MT} \begin{bmatrix} I'_{2N\times 2N} & \widetilde{Q}'_{a_1T} & \cdots & \widetilde{Q}'_{a_nT} & \widetilde{Q}'_{bT} & \widetilde{Q}'_{cT} \end{bmatrix}'. \quad (17)$$

The next result relates the solvability of the Riccati differential equations (14) with invertibility of the matrix $H(T)$.

Theorem 2. *[30, Theorem 7.1] For the N player finite horizon LQDG described by (12), the coupled Riccati differential equation (14) has a solution over the interval $[0,T]$ if and only if the matrix $H(T)$ is invertible.*

Remark 4. From [30, Proposition 7.6], if the following Riccati differential equations

$$\dot{K}_i = -Q_i + K_iS_iK_i,\ K_i(T) = \widetilde{Q}_{iT} \quad (18)$$

have a symmetrical solution K_i for each player i over the interval $[0,T]$, and together with the invertibility of the matrix $H(T)$ it can be shown that the Riccati differential equations (14) has a solution on $[0,T]$.

Using the above, the solution of the state equation (16) is given by

$$X(t) = \begin{bmatrix} I_{2N\times 2N} & 0 & \cdots & 0 \end{bmatrix} e^{M(t-T)} \begin{bmatrix} I & \widetilde{Q}'_{a_{1T}} \cdots \widetilde{Q}'_{a_nT} & \widetilde{Q}'_{b_T} & \widetilde{Q}'_{c_T} \end{bmatrix}' H^{-1}(T)X_0. \quad (19)$$

Remark 5. The horizon length must be selected such that the matrix $H(T)$ given by (17) is invertible. Later, in Lemma 1 we show that there exists an upper bound on T which ensures a certain behavior of the players.

IV. Switching analysis using receding horizon approach

In the previous section, we have analyzed the situation where the interactions of the players are fixed for a duration $[0,T]$ in one of the modes, and computed the outcome of the game as the open-loop Nash equilibrium strategies. Now, to allow for switching between the modes and for players to adapt their strategies, the open-loop Nash equilibrium solution is augmented with the *receding horizon* or moving horizon approach. In this method, every player computes the open-loop Nash equilibrium at each instant of time and implements the computed strategy for only one-time step. Players then repeat the procedure until the termination criteria are met while updating any change in the mode of the game (by the defender) and the closest target (by the attacker). In [31], the concept of moving horizon strategies in differential games was introduced. In [2], these strategies were studied in TAD differential games.

We now present the receding horizon approach for the N player game. We consider the policy or strategy time instants

$t_k = k\delta,\ k = 0,1,2,\cdots,$ with $t_0 = 0$ and $0 < \delta \ll T$. At any time instant t_k, using $X(t_k)$ as the initial state, players evaluate the open-loop Nash equilibrium control strategy over the planning horizon $[t_k, t_k+T]$, that is, players $i \in \mathscr{P}$ minimize the performance indices given by

$$J_i^{\mathrm{RH}}(u_{a_1}(.), u_{a_2}(.), \cdots, u_{a_n}(.), u_b(.), u_c(.); t_k, X(t_k)) = G_i(t_k+T) + \int_{t_k}^{t_k+T} L_i(\tau) d\tau. \quad (20)$$

The open-loop Nash equilibrium strategy of player i over the interval $[t_k, t_k+T]$ is obtained from (15). However, the open-loop Nash equilibrium strategies are implemented only for the period $[t_k, t_{k+1})$, and the receding horizon Nash control for player i at time $t \in [t_k, t_{k+1})$ with the initial state variable $X(t_k)$ is then given by

$$u_i^{\mathrm{RH}}(t; X(t_k)) = -R_i^{-1} B_i' P_i(t-t_k) \Phi(t-t_k, 0) X(t_k). \quad (21)$$

The state variable at time instant t_{k+1} is obtained from (19) as

$$X(t_{k+1}) = \begin{bmatrix} I_{2N\times 2N} & 0 & \cdots & 0 \end{bmatrix} e^{M(t_{k+1}-T)} \begin{bmatrix} I\ \widetilde{Q}'_{a_1T} \cdots \widetilde{Q}'_{a_nT}\ \widetilde{Q}'_{bT}\ \widetilde{Q}'_{cT} \end{bmatrix}' H^{-1}(T) X(t_k). \quad (22)$$

Next, at the time instant t_{k+1} the LQDG described by the objectives (20) and the dynamics (3) is solved by setting $t_k \to t_{k+1}$ and $X(t_k) \to X(t_{k+1})$ for the duration $[t_{k+1}, t_{k+1}+T]$. Again, the open-loop Nash equilibrium strategies of players, obtained similarly as (21), are implemented only for the period $[t_{k+1}, t_{k+2})$ to obtain the state variable $X(t_{k+2})$. This procedure is repeated again till the game termination criteria are met.

A. Termination criteria

To define the termination criterion, we denote the positive real numbers σ_b and σ_c as the capture radii of defender and attacker respectively. In the rescue mode, the game terminates when the defender rescues all the targets, that is, $||X_a(t) - X_b(t)||_2 \le \sigma_b$ for all $a \in \mathscr{A}$, or when attacker captures at least one target, that is, $||X_c(t) - X_a(t)||_2 \le \sigma_c$ for at least one $a \in \mathscr{A}$. Similarly, in the interception mode, the game terminates when the defender intercepts the attacker, that is, when $||X_b(t) - X_c(t)||_2 \le \sigma_b$ or when the attacker captures a target, that is, $||X_c(t) - X_a(t)||_2 \le \sigma_c$ for at least one $a \in \mathscr{A}$.

B. Target update by the attacker

The closest target pursued by the attacker can change with time requiring the attacker to update the closest target as the game proceeds in time. To incorporate this feature (F1), we assume that a_m is the closest target to the attacker at time instant t_k, that is,

$$a_m := \arg\min_{a\in\mathscr{A}} ||X_a(t_k) - X_c(t_k)||_2, \quad (23)$$

then the attacker pursues the target a_m and plays the game described by the objectives (20) to obtain the open-loop Nash equilibrium strategies for the duration $[t_k, t_k+T]$. The attacker implements these strategies only for the duration $[t_k, t_{k+1})$. Then at the time instant t_{k+1} the attacker reevaluates the closest target to pursue, using (23), and computes the open-loop Nash equilibrium strategy for the duration $[t_{k+1}, t_{k+1}+T]$, and implements it for the duration $[t_{k+1}, t_{k+2})$. This process is repeated until the termination criterion is met.

Remark 6. It is possible that at the time instant t_k two or more targets could be at the closest distance to the attacker. To handle such a scenario, we assume that the attacker moves towards the target that is farthest from the defender. If these targets are equidistant from the defender then we assume that the attacker chooses the target with lower index.

C. Operational mode switch by the defender

The defender can switch autonomously from rescue mode to interception mode and vice-versa, depending upon the state of the system; see feature (F2). We assume that the defender uses a switching function $\Psi : \mathbb{R}^n \to \mathbb{R}$ based on which the mode switching is realized at t_k, that is,

$$(\alpha_R, \alpha_I) = \begin{cases} (0,1), & \Psi(X(t_k)) \le 0 \\ (1,0), & \Psi(X(t_k)) > 0. \end{cases} \quad (24)$$

This implies, at time t_k, if $\Psi(X(t_k)) \le 0$ then the parameters (α_R, α_I) in the objective functions (20) are set to $(0,1)$ (interception mode), and if $\Psi(X(t_k)) > 0$ they are set to $(1,0)$ (rescue mode). Once the operational mode is decided by the defender at time instant t_k, the open-loop Nash equilibrium strategies (21) are evaluated for the duration $[t_k, t_k+T]$ and implemented for the duration $[t_k, t_{k+1})$. Then, at the next time instant t_{k+1} the same procedure is repeated until termination criterion is met. As the state information is available to all the players, a distance-based criterion is a natural choice for the switching function. In this paper, we consider the following switching function

$$\Psi(X(t)) := ||X_c(t) - X_{a_m}(t)||_2 - \kappa\sigma_c. \quad (25)$$

This implies, when the distance between the attacker and the minimum distance target to the attacker, evaluated at time instant t_k, is less than or equal to $\kappa\sigma_c > 0$ with $\kappa \ge 1$, then the defender sets the operational mode as interception mode for the duration $[t_k, t_{k+1})$. Here, the parameter κ indicates the level of alertness of the defender. In other words, a highly alert defender reacts early to an attacker who is approaching the target by switching from rescue mode to interception mode. The defender can implement these operational modes in coordination with the targets whenever a change in the sign of switching function (25) is observed, and this addresses feature (F3).

Remark 7. We emphasize that in the receding horizon approach the interactions between the players remain fixed between the time instants t_k and t_{k+1}. This implies that players only require state information at the time instants $t_k,\ k = 0,1,2,\cdots$ to implement the mode-dependent switching strategies.

Remark 8. From (6-11), the parameters (α_R, α_I) appear only in the objectives of the defender and the targets and not the attacker. This implies that in our setting the attacker is oblivious to mode switching of the defender.

Remark 9. In (20) there is a tacit assumption that all the players use the same horizon length T while synthesizing their receding horizon strategies.

D. Algorithm

The receding horizon approach for obtaining switching strategies is presented in Algorithm 1. The game is considered to terminate if the termination criteria mentioned in subsection IV-A are satisfied. We set the variable `tflag` equal to 1 to indicate termination of the game otherwise `tflag` is set as 0. Using initial state information $X_i(t_0)$, the attacker updates the closest target information at the time instant t_k; see steps 3-13. In step 7 the function $\texttt{minindex}(.)$ provides the element of the set of targets which has the minimum index. Based on switching function (25), the defender updates the operational mode autonomously at time instant t_k; see steps 14-19. The open-loop Nash equilibrium strategies are computed over the duration $[t_k, t_k+T]$; see steps 20-21. These strategies are implemented during the interval $[t_k, t_{k+1})$ to obtain the state variable at $X(t_{k+1})$; see steps 22-24. At every time instant t_k the termination conditions are verified; see steps 25-40. The next time instant is obtained by setting k as $k+1$; see step 41. At step 2, it is verified if the game is terminated or not by checking the `tflag` variable. If the game does not terminate, then the game continues with the state information $X(t_{k+1})$ which is computed already at step 24 in the previous iteration.

Algorithm 1: Synthesis of switching strategies using receding horizon approach

Data: R_i, $\widetilde{Q}_i$ and $\widetilde{Q}_{iT}$, $i \in \mathscr{P}$, κ, σ_b, σ_c, δ, T such that $H(T)$ is invertible.
Input: Initial locations $X_i(0)$, $i \in \mathscr{P}$
Output: Outcome of the game: Rescue of all targets (or) capture of a target by the attacker (or) interception of the attacker by the defender.

```
1  Initialize k = 0, t_0 = 0, X(t_0) = X(0) and the termination flag tflag = 0
   /* Iterate till the termination of the game */
2  while tflag=0 do
3      𝒯 := argmin_{a∈𝒜} ||X_a(t_k) − X_c(t_k)||_2   // Attacker choosing the minimum distance target
4      if |𝒯| > 1 then
5          𝒮 := argmax_{a∈𝒯} ||X_a(t_k) − X_b(t_k)||_2
6          if |𝒮| > 1 then
7              a_m = minindex(𝒮)
8          else
9              a_m = 𝒮
10         end
11     else
12         a_m = 𝒯
13     end
14     if Ψ(X(t_k)) ≤ 0          // Game mode using (24)
15      then
16         (α_R, α_I) = (0,1)     // Interception mode
17     else
18         (α_R, α_I) = (1,0)     // Rescue mode
19     end
20     Update Q̃_i, Q̃_iT and compute H(T) using (17)
21     Solve the Riccati differential equation (14) to obtain P_i(t_k) for all i ∈ 𝒫
22     Set t_{k+1} = t_k + δ
23     Implement the open-loop Nash equilibrium strategies u_i^RH(t, X(t_k)), t ∈ [t_k, t_{k+1}) using (21)
24     Compute X(t_{k+1}) using (19)
25     if (α_R, α_I) = (1,0) then
26         if ||X_a(t_k) − X_b(t_k)||_2 ≤ σ_b ∀a ∈ 𝒜 for t < T then
27             Rescue of all targets by the defender
28             Set tflag=1
29         end
30     end
31     if (α_R, α_I) = (0,1) then
32         if ||X_b(t_k) − X_c(t_k)||_2 ≤ σ_b for t < T then
33             Interception of the attacker by the defender
34             Set tflag=1
35         end
36     end
37     if ||X_{a_m}(t_k) − X_c(t_k)||_2 ≤ σ_c then
38         Capture of the target by the attacker
39         Set tflag=1
40     end
41     k ← k + 1
42 end
```

V. Analysis of the switching strategies

In this section, we analyze the switching strategies obtained through receding horizon approach, and derive results related to the trajectories of the players.

Assumption 1. The matrices $Q_{ij} = Q_{ijT} = I$ for all $i, j \in \mathscr{P}$, $i \neq j$ and $R_i = r_i I$, $r_i > 0$, for $i \in \mathscr{P}$.

The above assumption implies that players minimize or maximize their Euclidean distances with other players, and the penalties on the control efforts in x and y orientations are treated equally. In this mode, the attacker c is in direct conflict with its closest target a_m, and the other targets in $\mathscr{A} \backslash a_m$ and the defender b react to the outcome of this interaction. Based on this observation we have the following result.

Theorem 3. *Let Assumption 1 holds true. Let t_k be the time instant when the game switches to the interception mode. Then, using the receding horizon strategies (21), the attacker c and its closest target a_m move on the straight line joining their locations, evaluated at t_k, for the duration $[t_k, t_{k+1})$.*

Proof. We consider the interaction between the players $\{a_m, a, b, c\}$. The Riccati differential equation (14) associated with player $i \in \{a_m, a, b, c\}$ is given by

$$\dot{P}_i = -\widetilde{Q}_i + P_i(S_{a_m}P_{a_m} + S_a P_a + S_b P_b + S_c P_c), \tag{26}$$

where $P_i(t_k+T) = \widetilde{Q}_{iT}$. In the interception mode, the matrices entering the objective functions are $\widetilde{Q}_{a_m} = \widetilde{Q}_{a_m T} = \begin{bmatrix} -I & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & -I \end{bmatrix}$; $\widetilde{Q}_a = \widetilde{Q}_{aT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -I & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I \end{bmatrix}$; $\widetilde{Q}_b = \widetilde{Q}_{bT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & -I & I \end{bmatrix}$ and $\widetilde{Q}_c = \widetilde{Q}_{cT} = \begin{bmatrix} I & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & I \end{bmatrix}$. Then, it follows immediately that

$$\dot{P}_{a_m} + \dot{P}_c = (P_{a_m} + P_c)(S_{a_m}P_{a_m} + S_a P_a + S_b P_b + S_c P_c),$$

with $P_{a_m}(t_k+T)+P_c(t_k+T)=0$. This implies that

$$P_{a_m}(t)+P_c(t)=0 \Rightarrow P_{a_m}(t)=-P_c(t),\ t\in[t_k,t_k+T]. \quad (27)$$

We partition the matrix $P_i(t)$ for $i\in\{a_m,a,b,c\}$ as

$$P_i(t)=\begin{bmatrix} P_i^{11} & P_i^{12} & P_i^{13} & P_i^{14} \\ P_i^{21} & P_i^{22} & P_i^{23} & P_i^{24} \\ P_i^{31} & P_i^{32} & P_i^{33} & P_i^{34} \\ P_i^{41} & P_i^{42} & P_i^{43} & P_i^{44} \end{bmatrix}. \quad (28)$$

We denote by $\Gamma_1(t) := S_{a_m}P_{a_m}(t)+S_aP_a(t)+S_bP_b(t)+S_cP_c(t)$. Substituting for $S_i=B_iR_i^{-1}B_i'$ and $R_i=r_iI$ for $i\in\{a_m,a,b,c\}$ we obtain

$$\Gamma_1(t)=\begin{bmatrix} r_{a_m}^{-1}P_{a_m}^{11} & r_{a_m}^{-1}P_{a_m}^{12} & r_{a_m}^{-1}P_{a_m}^{13} & r_{a_m}^{-1}P_{a_m}^{14} \\ r_a^{-1}P_a^{21} & r_a^{-1}P_a^{22} & r_a^{-1}P_a^{23} & r_a^{-1}P_a^{24} \\ r_b^{-1}P_b^{31} & r_b^{-1}P_b^{32} & r_b^{-1}P_b^{33} & r_b^{-1}P_b^{34} \\ -r_c^{-1}P_{a_m}^{41} & -r_c^{-1}P_{a_m}^{42} & -r_c^{-1}P_{a_m}^{43} & -r_c^{-1}P_{a_m}^{44} \end{bmatrix}. \quad (29)$$

Using (28) in (26) for $i=a_m$, and pre-multiplying with the matrix $\begin{bmatrix} I & I & I & I\end{bmatrix}$ we obtain

$$\begin{aligned}&\begin{bmatrix}\sum_{l=1}^4\dot P_{a_m}^{l1} & \sum_{l=1}^4\dot P_{a_m}^{l2} & \sum_{l=1}^4\dot P_{a_m}^{l3} & \sum_{l=1}^4\dot P_{a_m}^{l4}\end{bmatrix}\\ &=\begin{bmatrix}\sum_{l=1}^4 P_{a_m}^{l1} & \sum_{l=1}^4 P_{a_m}^{l2} & \sum_{l=1}^4 P_{a_m}^{l3} & \sum_{l=1}^4 P_{a_m}^{l4}\end{bmatrix}\Gamma_1(t), \quad (30)\end{aligned}$$

where $\sum_{l=1}^4 P_{a_m}^{l1}(t_k+T)=\sum_{l=1}^4 P_{a_m}^{l2}(t_k+T)=\sum_{l=1}^4 P_{a_m}^{l3}(t_k+T)=\sum_{l=1}^4 P_{a_m}^{l4}(t_k+T)=0$. This implies that

$$P_{a_m}^{1j}(t)+P_{a_m}^{2j}(t)+P_{a_m}^{3j}(t)+P_{a_m}^{4j}(t)=0 \quad (31)$$

for all $t\in[t_k,t_k+T]$ for every $j\in\{1,2,3,4\}$. Again, pre-multiplying (26) with $\begin{bmatrix} I & 0 & 0 & I\end{bmatrix}$ and repeating the same analysis as before we obtain

$$P_{a_m}^{1j}(t)+P_{a_m}^{4j}(t)=0 \Rightarrow P_{a_m}^{1j}(t)=-P_{a_m}^{4j}(t) \quad (32)$$

for all $t\in[t_k,t_k+T]$ for every $j\in\{1,2,3,4\}$. Using (32) in (31) we have

$$P_{a_m}^{2j}(t)+P_{a_m}^{3j}(t)=0 \Rightarrow P_{a_m}^{2j}(t)=-P_{a_m}^{3j}(t) \quad (33)$$

for all $t\in[t_k,t_k+T]$ for every $j\in\{1,2,3,4\}$. Again, using (28) in (26) for $i=a_m$, and post-multiplying with the matrix $\begin{bmatrix} I & I & I & I\end{bmatrix}'$ we obtain

$$\begin{bmatrix}\sum_{l=1}^4\dot P_{a_m}^{1l}\\ \sum_{l=1}^4\dot P_{a_m}^{2l}\\ \sum_{l=1}^4\dot P_{a_m}^{3l}\\ \sum_{l=1}^4\dot P_{a_m}^{4l}\end{bmatrix}=P_{a_m}\begin{bmatrix} r_{a_m}^{-1}\sum_{l=1}^4 P_{a_m}^{1l}\\ r_a^{-1}\sum_{l=1}^4 P_a^{2l}\\ r_b^{-1}\sum_{l=1}^4 P_b^{3l}\\ -r_c^{-1}\sum_{l=1}^4 P_{a_m}^{4l}\end{bmatrix},\ \begin{bmatrix}\sum_{l=1}^4 P_{a_m}^{1l}(t_k+T)\\ \sum_{l=1}^4 P_{a_m}^{2l}(t_k+T)\\ \sum_{l=1}^4 P_{a_m}^{3l}(t_k+T)\\ \sum_{l=1}^4 P_{a_m}^{4l}(t_k+T)\end{bmatrix}=0 \quad (34)$$

Repeating the above exercise for the matrices P_a and P_c and then rearranging terms we obtain

$$\begin{bmatrix}\sum_{l=1}^4\dot P_{a_m}^{1l}\\ \sum_{l=1}^4\dot P_a^{2l}\\ \sum_{l=1}^4\dot P_b^{3l}\\ \sum_{l=1}^4\dot P_{a_m}^{4l}\end{bmatrix}=\Xi(t)\begin{bmatrix}\sum_{l=1}^4 P_{a_m}^{1l}\\ \sum_{l=1}^4 P_a^{2l}\\ \sum_{l=1}^4 P_b^{3l}\\ \sum_{l=1}^4 P_{a_m}^{4l}\end{bmatrix},\ \begin{bmatrix}\sum_{l=1}^4 P_{a_m}^{1l}(t_k+T)\\ \sum_{l=1}^4 P_a^{2l}(t_k+T)\\ \sum_{l=1}^4 P_b^{3l}(t_k+T)\\ \sum_{l=1}^4 P_{a_m}^{4l}(t_k+T)\end{bmatrix}=0,$$

where

$$\Xi(t)=\begin{bmatrix} r_{a_m}^{-1}P_{a_m}^{11} & r_a^{-1}P_a^{12} & r_b^{-1}P_b^{13} & -r_c^{-1}P_{a_m}^{14} \\ r_{a_m}^{-1}P_{a_m}^{21} & r_a^{-1}P_a^{22} & r_b^{-1}P_b^{23} & -r_c^{-1}P_{a_m}^{24} \\ r_{a_m}^{-1}P_{a_m}^{31} & r_a^{-1}P_a^{32} & r_b^{-1}P_b^{33} & -r_c^{-1}P_{a_m}^{34} \\ r_{a_m}^{-1}P_{a_m}^{41} & r_a^{-1}P_a^{42} & r_b^{-1}P_b^{43} & -r_c^{-1}P_{a_m}^{44} \end{bmatrix}. \quad (35)$$

This implies that

$$\sum_{l=1}^4 P_{a_m}^{1l}(t)=\sum_{l=1}^4 P_a^{2l}(t)=\sum_{l=1}^4 P_b^{3l}(t)=\sum_{l=1}^4 P_{a_m}^{4l}(t)=0 \quad (36)$$

for all $t\in[t_k,t_k+T]$. Next, using (36) in (34) we also have that

$$\sum_{l=1}^4 P_{a_m}^{2l}(t)=\sum_{l=1}^4 P_{a_m}^{3l}(t)=0 \quad (37)$$

for all $t\in[t_k,t_k+T]$. Next, using (32) we analyze the elements $P_{a_m}^{12}$ and $P_{a_m}^{13}$

$$\begin{aligned}\dot P_{a_m}^{12}&=r_{a_m}^{-1}P_{a_m}^{11}P_{a_m}^{12}+r_a^{-1}P_{a_m}^{12}P_a^{22}+r_b^{-1}P_{a_m}^{13}P_b^{32}-r_c^{-1}P_{a_m}^{14}P_{a_m}^{42}\\ &=\left(r_{a_m}^{-1}P_{a_m}^{11}+r_c^{-1}P_{a_m}^{14}\right)P_{a_m}^{12}+r_a^{-1}P_{a_m}^{12}P_a^{22}+r_b^{-1}P_{a_m}^{13}P_b^{32}\\ \dot P_{a_m}^{13}&=r_{a_m}^{-1}P_{a_m}^{11}P_{a_m}^{13}+r_a^{-1}P_{a_m}^{12}P_a^{23}+r_b^{-1}P_{a_m}^{13}P_b^{33}-r_c^{-1}P_{a_m}^{14}P_{a_m}^{43}\\ &=\left(r_{a_m}^{-1}P_{a_m}^{11}+r_c^{-1}P_{a_m}^{14}\right)P_{a_m}^{13}+r_a^{-1}P_{a_m}^{12}P_a^{23}+r_b^{-1}P_{a_m}^{13}P_b^{33}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}\dot P_{a_m}^{12} & \dot P_{a_m}^{13}\end{bmatrix}&=\left(r_{a_m}^{-1}P_{a_m}^{11}+r_c^{-1}P_{a_m}^{14}\right)\begin{bmatrix}P_{a_m}^{12} & P_{a_m}^{13}\end{bmatrix}\\ &\quad+\begin{bmatrix}P_{a_m}^{12} & P_{a_m}^{13}\end{bmatrix}\begin{bmatrix} r_a^{-1}P_a^{22} & r_a^{-1}P_a^{23}\\ r_b^{-1}P_b^{32} & r_b^{-1}P_b^{33}\end{bmatrix}.\end{aligned}$$

The terminal conditions are $P_{a_m}^{12}(t_k+T)=0$ and $P_{a_m}^{13}(t_k+T)=0$. From the matrix variation of constants formula [32, Theorem 1, pg. 59] it follows immediately that $P_{a_m}^{12}(t)=P_{a_m}^{13}(t)=0$ for all $t\in[t_k,t_k+T]$. Then from (32) we have

$$P_{a_m}^{12}(t)=P_{a_m}^{13}(t)=P_{a_m}^{42}(t)=P_{a_m}^{43}(t)=0,\ t\in[t_k,t_k+T]. \quad (38)$$

Next, using (38) in (26) with $i=a_m$ we have

$$\begin{bmatrix}\dot P_{a_m}^{22} & \dot P_{a_m}^{23}\end{bmatrix}=\begin{bmatrix}P_{a_m}^{22} & P_{a_m}^{23}\end{bmatrix}\begin{bmatrix} r_a^{-1}P_a^{22} & r_a^{-1}P_a^{23}\\ r_b^{-1}P_b^{32} & r_b^{-1}P_b^{33}\end{bmatrix} \quad (39)$$

with $P_{a_m}^{22}(t_k+T)=P_{a_m}^{23}(t_k+T)=0$. This coupled with (33) implies

$$P_{a_m}^{22}(t)=P_{a_m}^{23}(t)=P_{a_m}^{32}(t)=P_{a_m}^{33}(t)=0,\ t\in[t_k,t_k+T]. \quad (40)$$

Again, using (40) in (26) with $i=a_m$ we have

$$\begin{bmatrix}\dot P_{a_m}^{21} & \dot P_{a_m}^{24}\end{bmatrix}=\begin{bmatrix}P_{a_m}^{21} & P_{a_m}^{24}\end{bmatrix}\begin{bmatrix} r_{a_m}^{-1}P_{a_m}^{11} & r_{a_m}^{-1}P_{a_m}^{14}\\ -r_c^{-1}P_{a_m}^{41} & -r_c^{-1}P_{a_m}^{44}\end{bmatrix} \quad (41)$$

with $P_{a_m}^{21}(t_k+T)=P_{a_m}^{24}(t_k+T)=0$. This coupled with (33) implies

$$P_{a_m}^{21}(t)=P_{a_m}^{24}(t)=P_{a_m}^{31}(t)=P_{a_m}^{34}(t)=0,\ t\in[t_k,t_k+T]. \quad (42)$$

Thus the structure of matrix P_{a_m} is given by

$$P_{a_m}(t)=-P_c(t)=\begin{bmatrix}-K(t) & 0 & 0 & K(t)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ K(t) & 0 & 0 & -K(t)\end{bmatrix}. \quad (43)$$

where $K(t)=\begin{bmatrix} k_1(t) & k_2(t)\\ k_3(t) & k_4(t)\end{bmatrix}_{2\times 2}$, $k_i(t)\in\mathbb{R}$, $i=1,2,3,4$. Next, writing $P_{a_m}^{11}=-K$ in (26) we get

$$\dot K=-(r_{a_m}^{-1}-r_c^{-1})KK-I,\ K(t_k+T)=I. \quad (44)$$

Now, expanding K we have

$$\begin{aligned}\dot{k}_1 &= -(r_{a_m}^{-1} - r_c^{-1})(k_1^2 + k_2 k_3) - 1,\ k_1(t_k+T) = 1\\ \dot{k}_2 &= -(r_{a_m}^{-1} - r_c^{-1})(k_1 + k_4)k_2,\ k_2(t_k+T) = 0\\ \dot{k}_3 &= -(r_{a_m}^{-1} - r_c^{-1})(k_1 + k_4)k_3,\ k_3(t_k+T) = 0\\ \dot{k}_4 &= -(r_{a_m}^{-1} - r_c^{-1})(k_3 k_2 + k_4^2) - 1,\ k_4(t_k+T) = 1.\end{aligned}$$

Notice, $k_2(t)$ and $k_3(t)$ are solutions of the differential equation

$$\dot{\gamma} = -(r_{a_m}^{-1} - r_c^{-1})(k_1 + k_4)\gamma,\ \gamma(t_k+T) = 0.$$

This implies that $k_2(t) = k_3(t) = 0$ for all $t \in [t_k, t_k+T]$. Next, $k_1(t)$ and $k_4(t)$ satisfy the differential equation

$$\dot{\zeta}_1 = -(r_{a_m}^{-1} - r_c^{-1})\zeta_1^2 - 1,\ \zeta_1(t_k+T) = 1. \tag{45}$$

So, we have that $k_1(t) = k_4(t) = \zeta_1(t)$ for all $t \in [t_k, t_k+T]$. So, we have

$$P_{a_m}(t) = -P_c(t) = \zeta_1(t)\begin{bmatrix} -I & 0 & 0 & I\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ I & 0 & 0 & -I\end{bmatrix}. \tag{46}$$

Now, using the open-loop Nash equilibrium strateiges (15) in the game with players $\{a_m, a, b, c\}$ the state variables of the target a_m and the attacker c are given by

$$\begin{bmatrix}\dot{X}_{a_m}\\ \dot{X}_c\end{bmatrix} = -\zeta_1(t)\begin{bmatrix} R_{a_m}^{-1}\\ R_c^{-1}\end{bmatrix}(X_c - X_{a_m}).$$

Using the above we have

$$\begin{aligned}\dot{X}_c - \dot{X}_{a_m} &= -\zeta_1(t)(R_c^{-1} - R_{a_m}^{-1})(X_c - X_{a_m})\\ &= \left(\frac{r_c - r_{a_m}}{r_c r_{a_m}}\right)\zeta_1(t)(X_c - X_{a_m}).\end{aligned} \tag{47}$$

Representing the x and y co-ordinates of $X_c - X_{a_m}$ as $z_1 := x_c - x_{a_m}$ and $z_2 := y_c - y_{a_m}$, the above equation can be written as

$$\dot{z}_1 = \left(\frac{r_c - r_{a_m}}{r_c r_{a_m}}\right)\zeta_1(t) z_1,\ \dot{z}_2 = \left(\frac{r_c - r_{a_m}}{r_c r_{a_m}}\right)\zeta_1(t) z_2. \tag{48}$$

Slope of the line joining the attacker c and the target a_m at time t is given by $s_1(t) = \frac{y_c(t) - y_{a_m}(t)}{x_c(t) - x_{a_m}(t)} = \frac{z_2(t)}{z_1(t)},\ z_1(t) \neq 0$. From (48) we have that when $z_1(t_k) \neq 0$ then $z_1(t) \neq 0$ for all $t \in [t_k, t_k+T]$. The time derivative of the slope $s_1(t)$ results in

$$\begin{aligned}\dot{s}_1(t) &= \frac{\dot{z}_2(t) z_1(t) - \dot{z}_1(t) z_2(t)}{z_1^2(t)}\\ &= \left(\frac{r_c - r_{a_m}}{r_{a_m} r_c}\right)\left(\frac{z_1(t) z_2(t) - z_1(t) z_2(t)}{z_1^2(t)}\right)\zeta_1(t) = 0.\end{aligned}$$

Clearly, this implies when $x_c(t_k) \neq x_{a_m}(t_k)$ the slope $s_1(t) = s_1(t_k)$ for all $t \in [t_k, t_{k+1})$. When $z_1(t_k) = x_c(t_k) - x_{a_m}(t_k) = 0$ then $x_c(t) = x_{a_m}(t)$ for all t, this implies the attacker c and the target a_m continue along the y-axis for all $t \in [t_k, t_{k+1})$. ■

Remark 10. In the proof of the Theorem 3 it is sufficient to consider the interaction between the four players $\{a_m, a, b, c\}$. This is because the defender b and the targets $a \in \mathscr{A}\backslash\{a_m\}$ are responding to the direct interaction of the attacker c and its minimum distance target a_m.

Next, we have the following result to infer about the geometric structure of trajectories of the targets $a \in \mathscr{A}\backslash a_m$.

Theorem 4. *Let Assumption 1 holds true. Let t_k be the time instant when the game switches to the interception mode. Let $r_a = r_{a_i}$ for all $a, a_i \in \mathscr{A}$ and $a_i \neq a$. Then the line joining the targets a_m and $a \in \mathscr{A}\backslash\{a_m\}$ evolves with a constant slope for the time duration $[t_k, t_{k+1})$.*

Proof. Using the open loop Nash controls, (43) and (29), the state vector is written as:

$$\begin{bmatrix}\dot{X}_{a_m}\\ \dot{X}_a\\ \dot{X}_b\\ \dot{X}_c\end{bmatrix} = -\begin{bmatrix} -r_{a_m}^{-1}K(t) & 0 & 0 & r_{a_m}^{-1}K(t)\\ r_a^{-1}P_a^{21} & r_a^{-1}P_a^{22} & r_a^{-1}P_a^{23} & r_a^{-1}P_a^{24}\\ r_b^{-1}P_b^{31} & r_b^{-1}P_b^{32} & r_b^{-1}P_b^{33} & r_b^{-1}P_b^{34}\\ -r_c^{-1}K(t) & 0 & 0 & r_c^{-1}K(t)\end{bmatrix}\begin{bmatrix} X_{a_m}\\ X_a\\ X_b\\ X_c\end{bmatrix}$$

Then the position vectors of a_m, a and c satisfy

$$\begin{aligned}\dot{X}_{a_m} &= r_{a_m}^{-1}K(t)(X_{a_m} - X_c)\\ \dot{X}_a &= -r_a^{-1}[P_a^{21}X_{a_m} + P_a^{22}X_a + P_a^{23}X_b + P_a^{24}X_c]\\ \dot{X}_c &= r_c^{-1}K(t)(X_{a_m} - X_c)\end{aligned}$$

Using the above, we have

$$\begin{aligned}\dot{X}_{a_m} - \dot{X}_a &= r_{a_m}^{-1}K(t)(X_{a_m} - X_c)\\ &\quad + r_a^{-1}[P_a^{21}X_{a_m} + P_a^{22}X_a + P_a^{23}X_b + P_a^{24}X_c].\end{aligned} \tag{49}$$

Next, using (26) and (28) for $i = a$, we have

$$\begin{aligned}\dot{P}_a^{23} &= r_{a_m}^{-1}P_a^{21}P_{a_m}^{13} + r_a^{-1}P_a^{22}P_a^{23}\\ &\quad + r_b^{-1}P_a^{23}P_b^{33} - r_c^{-1}P_a^{24}P_{a_m}^{43}, P_a^{23}(t_k+T) = 0\end{aligned} \tag{50}$$

Using (43), we can further reduce (50) to the following:

$$\dot{P}_a^{23} = (r_a^{-1}P_a^{22})P_a^{23} + P_a^{23}(r_b^{-1}P_b^{33}),\ P_a^{23}(t_k+T) = 0 \tag{51}$$

From the matrix variation of constants formula [32, Theorem 1, pg. 59] it follows immediately that

$$P_a^{23}(t) = 0,\ t \in [t_k, t_k+T]. \tag{52}$$

Similarly, (26) and applying (28) for $i = a, a_m$, we have

$$\begin{aligned}\dot{P}_a^{24} &= -I + r_{a_m}^{-1}P_a^{21}P_{a_m}^{14} + r_a^{-1}P_a^{22}P_a^{24} + r_b^{-1}P_a^{23}P_b^{34}\\ &\quad - r_c^{-1}P_a^{24}P_{a_m}^{44},\quad P_a^{24}(t_k+T) = I\end{aligned} \tag{53}$$

$$\begin{aligned}\dot{P}_{a_m}^{11} &= I + r_{a_m}^{-1}P_{a_m}^{11}P_{a_m}^{11} + r_a^{-1}P_{a_m}^{12}P_a^{21} + r_b^{-1}P_{a_m}^{13}P_b^{31}\\ &\quad - r_c^{-1}P_{a_m}^{14}P_{a_m}^{41},\ P_{a_m}^{11}(t_k+T) = -I\end{aligned} \tag{54}$$

Using $P_{a_m}^{11} = -K(t)$, $P_a^{23} = 0$, and $r_{a_m} = r_a$ we can further reduce (53) and (54) as

$$\begin{aligned}\dot{P}_a^{24} - \dot{K}(t) &= r_a^{-1}\left[P_a^{21}K(t) + P_a^{22}P_a^{24} + K(t)K(t)\right]\\ &\quad + r_c^{-1}\left[P_a^{24}K(t) - K(t)K(t)\right],\end{aligned} \tag{55}$$

with $P_a^{24}(t_k+T) - K(t_k+T) = 0$. From (36) and (52) we have

$$P_a^{21} + P_a^{22} + P_a^{23} + P_a^{24} = 0 \Rightarrow P_a^{21} = -P_a^{22} - P_a^{24}.$$

Using this in (55) we get

$$\begin{aligned}\dot{P}_a^{24} - \dot{K}(t) &= [r_a^{-1}P_a^{22}](P_a^{24} - K(t))\\ &\quad + (P_a^{24} - K(t))[(r_c^{-1} - r_a^{-1})K(t)],\end{aligned} \tag{56}$$

with $P_a^{24}(t_k+T)-K(t_k+T)=0$. Again, from the matrix variation of constants formula [32, Theorem 1, pg. 59] it follows immediately that

$$P_a^{24}=K(t),\ t\in[t_k,t_k+T]. \tag{57}$$

Using (28) for $i=a$ in (26), we have that P_a^{22} satisfies

$$\dot{P}_a^{22}=r_a^{-1}P_a^{22}P_a^{22}+I,\ P_a^{22}(t_k+T)=-I \tag{58}$$

We solve (58) using the same approach as in solving (44) to get $P_a^{22}(t)=\zeta_2(t)I$ where $\zeta_2(t)$ satisfies the differential equation

$$\dot{\zeta}_2(t)=r_a^{-1}\zeta_2^2(t)+1,\ \zeta_2(t_k+T)=-1. \tag{59}$$

Using the above, (49) can be written as

$$\begin{aligned}\dot{X}_{a_m}-\dot{X}_a&=r_a^{-1}K(t)(X_{a_m}-X_c)\\&\quad+r_a^{-1}\left[(-P_a^{22}-P_a^{23}-P_a^{24})X_{a_m}+P_a^{22}X_a+K(t)X_c\right]\\&=-r_a^{-1}P_a^{22}(X_{a_m}-X_a)\\&=-r_a^{-1}\zeta_2(t)\,(X_{a_m}-X_a)\end{aligned} \tag{60}$$

Representing the x and y co-ordinates of $X_{a_m}-X_a$ as $z_1:=x_{a_m}-x_a$ and $z_2:=y_{a_m}-y_a$, then (60) can be written as

$$\dot{z}_1=-r_a^{-1}\zeta_2 z_1,\ \dot{z}_2=-r_a^{-1}\zeta_2 z_2. \tag{61}$$

Slope of the line joining the target a_m and the target a at time t is given by $s_2(t)=\frac{y_{a_m}(t)-y_a(t)}{x_{a_m}(t)-x_a(t)}=\frac{z_2(t)}{z_1(t)}$, $z_1(t)\neq 0$. From (61) we have that when $z_1(t_k)\neq 0$ then $z_1(t)\neq 0$ for all $t\in[t_k,t_k+T]$. The time derivative of the slope $s_2(t)$ results in

$$\begin{aligned}\dot{s}_2(t)&=\frac{\dot{z}_2(t)z_1(t)-\dot{z}_1(t)z_2(t)}{z_1^2(t)}\\&=-r_a^{-1}\left(\frac{z_1(t)z_2(t)-z_1(t)z_2(t)}{z_1^2(t)}\right)\zeta_2(t)=0.\end{aligned}$$

Clearly, this implies when $x_{a_m}(t_k)\neq x_a(t_k)$ the slope $s_2(t)=s_2(t_k)$ for all $t\in[t_k,t_{k+1})$. When $z_1(t_k)=x_{a_m}(t_k)-x_a(t_k)=0$ then $x_{a_m}(t)=x_a(t)$ for all t, this implies the target a_m and the target a continue along the y-axis for all $t\in[t_k,t_{k+1})$. ■

Corollary 1. *Let Assumption 1 holds true. Let t_k be the time instant when the game switches to the interception mode. Let $r_{a_i}=r_{a_j}$ for all $a_i,a_j\in\mathscr{A}$ and $i\neq j$. Then the angle between the lines joining the attacker (c), the minimum distance target (a_m) and a target $a\in\mathscr{A}\backslash a_m$ remains constant for the duration $[t_k,t_{k+1})$.*

Proof. From Theorem 3 and Theorem 4 we know that slopes of the line joining the players c and a_m, and the line joining a_m and an $a\in\mathscr{A}$ remain constant during the execution period $[t_k,t_{k+1})$. The statement of the theorem follows immediately from this observation. ■

From Theorem 3, the attacker c and the target a_m move in a straight line till the next time instant t_{k+1}. Now, at t_{k+1} it is possible that the minimum distance target a_m (at time t_k) is no longer at a minimum distance to c as other targets in $\mathscr{A}\backslash a_m$ are trying to maximize their distance with c. In the following we derive conditions under which the target a_m remains to stay at a minimum distance to the attacker for the entire time duration $[t_k,t_{k+1})$. Towards this end, we provide some auxiliary results. Let us denote by $d_1:=X_{a_m}-X_c$ and $d_2:=X_{a_m}-X_a$. We have the following assumption on the decision horizon and the penalty parameters.

Lemma 1. *Let Assumption 1 holds true. Let t_k be the time instant when the game switches to the interception mode. Let us assume $d_1(t_k)\neq 0$ and $d_2(t_k)\neq 0$. Let $r_{a_i}=r_{a_j}$ for all $a_i,a_j\in\mathscr{A}$ and $i\neq j$, and the penalty parameters of a target $a\in\mathscr{A}$ and the attacker c satisfy the condition $0<\frac{r_a-r_c}{r_ar_c}<1$. Then, the distance between the attacker c and it's minimum distance target a_m, decreases with time for the time duration $[t_k,t_{k+1})$. Further, with $t_{k+1}-t_k=\delta$,*

1) *if the length of the planning horizon $T>0$ satisfies*

$$\begin{aligned}\sqrt{r_a}\left[\left(k-\frac{1}{2}\right)\pi-\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)\right]+\delta<\\ T<\sqrt{r_a}\left[k\pi-\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)\right],\ k\in\mathbb{Z},\end{aligned} \tag{62}$$

then distance between the targets a_m and $a\in\mathscr{A}\backslash\{a_m\}$, increases with time for the time duration $[t_k,t_{k+1})$,

2) *if the length of the planning horizon $T>0$ satisfies*

$$\begin{aligned}\sqrt{r_a}\left[k\pi-\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)\right]+\delta<\\ T<\sqrt{r_a}\left[\left(k+\frac{1}{2}\right)\pi-\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)\right],\ k\in\mathbb{Z},\end{aligned} \tag{63}$$

then distance between the targets a_m and $a\in\mathscr{A}\backslash\{a_m\}$, decreases with time for the time duration $[t_k,t_{k+1})$.

Proof. The equations (47) and (60) are given by

$$\dot{d}_1=-\frac{r_a-r_c}{r_ar_c}\zeta_1 d_1 \tag{64}$$

$$\dot{d}_2=-\frac{1}{r_a}\zeta_2 d_2. \tag{65}$$

Next, consider the functions $V_1(t)=\frac{1}{2}d_1'(t)d_1(t)$ and $V_2(t)=\frac{1}{2}d_2'(t)d_2(t)$ defined over the time duration $[t_k,t_{k+1})$. Clearly, $V_1(t)\geq 0$ and $V_2(t)\geq 0$ for all $t\in[t_k,t_k+T]$. Recall, $\zeta_1(t)$ and $\zeta_2(t)$ are obtained by solving (45) and (59).

$$\zeta_1(t)=\begin{cases}\sqrt{\frac{r_cr_a}{r_c-r_a}}\tan\left(\frac{t_k+T-t}{\sqrt{\frac{r_cr_a}{r_c-r_a}}}+\tan^{-1}\left(\sqrt{\frac{r_c-r_a}{r_cr_a}}\right)\right), & r_c>r_a\\ \sqrt{\frac{r_cr_a}{r_a-r_c}}\tanh\left(\frac{t_k+T-t}{\sqrt{\frac{r_cr_a}{r_a-r_c}}}+\tanh^{-1}\left(\sqrt{\frac{r_a-r_c}{r_cr_a}}\right)\right), & r_a>r_c\end{cases}$$

$$\zeta_2(t)=-\sqrt{r_a}\tan\left(\frac{t_k+T-t}{\sqrt{r_a}}+\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)\right),$$

where $\zeta_1(t_k+T)=1$ and $\zeta_2(t_k+T)=-1$. If the penalty parameters satisfy $0<\frac{r_a-r_c}{r_ar_c}<1$, it is easy to verify that $\zeta_1(t)>0$ for all $t\in[t_k,t_k+T]$; here, we used the fact that $\tanh(x)$ is defined for $|x|<1$. Taking the time derivative, we have $\dot{V}_1=\dot{d}_1'd_1=-\frac{r_a-r_c}{r_ar_c}\zeta_1 d_1'd_1<0$. This implies that the distance between the attacker c and the target a_m decreases strictly with time for the time duration $[t_k,t_{k+1})$. Next, when the planning horizon length $T>0$ satisfies the condition $\sqrt{r_a}k\pi<T+\sqrt{r_a}\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)<\sqrt{r_a}\left(k+\frac{1}{2}\right)\pi$ and $\sqrt{r_a}k\pi<T-\delta+\sqrt{r_a}\tan^{-1}\left(\frac{1}{\sqrt{r_a}}\right)<\sqrt{r_a}\left(k+\frac{1}{2}\right)\pi$ we have that $\zeta_2(t) < 0$ for all $t \in [t_k, t_{k+1}]$ with $t_{k+1} = t_k + \delta$. After rearranging these inequalities we obtain the condition (62). Taking the time derivative, we have $\dot{V}_2 = \dot{d}_2' d_2 = -\frac{1}{r_a}\zeta_2 d_2' d_2 > 0$ for $t \in [t_k, t_{k+1})$. This implies that the distance between the targets a_m and $a \in \mathscr{A} \backslash \{a_m\}$ increases strictly with time for the time duration $[t_k, t_{k+1})$. Using the same approach as above it is easy to verify that when $T > 0$ satisfies (63) then the distance between the targets a_m and $a \in \mathscr{A} \backslash \{a_m\}$ decreases strictly with time for the time duration $[t_k, t_{k+1})$. ■

Assumption 2. The targets are symmetric, that is, $r_{a_i} = r_{a_j}$ for all $a_i, a_j \in \mathscr{A}$ and $i \neq j$. The penalty parameters of a target $a \in \mathscr{A}$ and the attacker c satisfy the condition $0 < \frac{r_a - r_c}{r_a r_c} < 1$ and the policy horizon length $T > 0$ satisfies (62).

As an immediate consequence of Lemma 1 and Assumption 2, we have the following result.

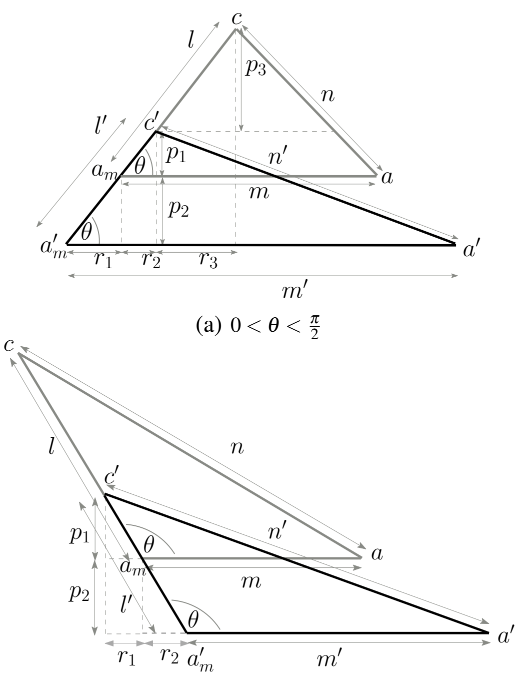


Fig. 2: Trajectories of players. Labels a_m, a and c are used to illustrate the position of minimum distance target, a target in $\mathscr{A} \backslash a_m$ and the attacker at time instant t_k. Labels a_m', a' and c' illustrate the position of the same players at a time instant $t \in (t_k, t_k + 1)$.

Theorem 5. *Let Assumptions 1 and 2 hold true. Let t_k be the time instant when the game switches to the interception mode. Then, the target which was at minimum distance to the attacker c at the time instant t_k continues to remain so at the time instant t_{k+1}.*

Proof. From Corollary 1, the angle between the lines joining the attacker c, the minimum distance target a_m, and the target $a \in \mathscr{A} \backslash a_m$ remains constant throughout the time duration $[t_k, t_{t+1})$. Let us denote this angle by θ. Firstly, we consider the case when $\theta \in \left(0, \frac{\pi}{2}\right)$ as illustrated in Figure 2a. Since a_m is the minimum distance target we have $n > l$. From Theorem 3 and Theorem 4 we have $l > l'$ and $m' > m$. Next, from the triangle $\triangle c a_m a$, we have $n^2 - l^2 = (p_1 + p_3)^2 + (m - (r_2 + r_3))^2 - (p_1 + p_3)^2 - (r_2 + r_3)^2 = m^2 - 2m(r_2 + r_3) = m(m - 2l\cos(\theta))$. As $n > l$ and $m > 0$, we have that $m > 2l\cos(\theta)$. Now, using the fact that $m' > m$ and $l > l'$, we get

$$m' > 2l'\cos(\theta). \tag{66}$$

From the triangle $\triangle c' a_m' a'$, we have $n'^2 - l'^2 (p_1 + p_2)^2 + (m' - (r_1 + r_2))^2 - (p_1 + p_2)^2 - (r_1 + r_2)^2 = m'^2 - 2m'(r_1 + r_2) = m'(m' - 2l'\cos(\theta))$. From (66) this implies $n' > l'$. Next, we consider the case when $\theta \in \left(\frac{\pi}{2}, \pi\right)$ as illustrated in Figure 2b. From the triangles $\triangle c' a_m a$ and $\triangle c' a_m' a'$ we have $n'^2 - l'^2 = (p_1 + p_2)^2 + (m + (r_1 + r_2))^2 - (p_1 + p_2)^2 - (r_1 + r_2)^2 = m(m + 2(r_1 + r_2)) > 0$.

Clearly, this implies $n' > l'$. When $\theta = \frac{\pi}{2}$, we have $r_1 + r_2 = 0$, then $n'^2 - l'^2 = m^2 > 0$. This implies that for $\theta \in (0, \pi)$ the statement of the theorem holds true. When $\theta = k\pi,\ k = 0, 1$, all the players lie on the same line and Lemma 1 provides the desired result. ■

Remark 11. Theorem 5 implies that the target which is at a minimum distance with the attacker, at time instant t_k, will remain so for all the time duration $t \in [t_k, t_k + 1)$. In other words, if the planning horizon length T is appropriately chosen as (62), then our assumption in Algorithm 1 that the attacker pursues the target a_m during the interval $[t_k, t_{k+1})$ without updating seems reasonable.

So far, we have analyzed the situation where the game enters the interception mode at time instant t_k. In the following theorem we study the nature of trajectories when the game enters rescue mode at t_k.

Theorem 6. *Let Assumption 1 holds true and let $r_{a_i} = r_{a_j}$ for all $a_i, a_j \in \mathscr{A}$ and $i \neq j$. Let t_k be the time instant when the game enters the rescue mode. Then, the target a_m which was at minimum distance to the attacker c at time instant t_k remains at constant distance and orientation with other targets $a \in \mathscr{A} \backslash a_m$ for the time duration $[t_k, t_{k+1})$.*

Proof. We consider the interaction between the players $\{a_m, a, b, c\}$. The Riccati differential equation (14) associated with player $i \in \{a_m, a, b, c\}$ is given by

$$\dot{P}_i = -\widetilde{Q}_i + P_i(S_{a_m} P_{a_m} + S_a P_a + S_b P_b + S_c P_c), \tag{67}$$

where $P_i(t_k + T) = \widetilde{Q}_{iT}$. In rescue mode, we have $\widetilde{Q}_{a_m} = \widetilde{Q}_{a_m T} = \begin{bmatrix} 0 & 0 & -I & I \\ 0 & 0 & 0 & 0 \\ -I & 0 & I & 0 \\ I & 0 & 0 & -I \end{bmatrix}$; $\widetilde{Q}_a = \widetilde{Q}_{aT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & I \\ 0 & -I & I & 0 \\ 0 & I & 0 & -I \end{bmatrix}$; $\widetilde{Q}_b = \widetilde{Q}_{bT} = \begin{bmatrix} I & 0 & -I & 0 \\ 0 & I & -I & 0 \\ -I & -I & 2I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\widetilde{Q}_c = \widetilde{Q}_{cT} = \begin{bmatrix} I & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & I \end{bmatrix}$. Using the open-loop Nash equilibrium controls, the state vector is written as

$$\dot{X}(t) = -(S_{a_m} P_{a_m} + S_a P_a + S_b P_b + S_c P_c) X(t).$$

We partition the matrix $P_i(t)$ for $i \in \{a_m, a, b, c\}$ similar to (28) to obtain

$$\begin{bmatrix} \dot{X}_{a_m} \\ \dot{X}_a \\ \dot{X}_b \\ \dot{X}_c \end{bmatrix} = -\begin{bmatrix} r_{a_m}^{-1} P_{a_m}^{11} & r_{a_m}^{-1} P_{a_m}^{12} & r_{a_m}^{-1} P_{a_m}^{13} & r_{a_m}^{-1} P_{a_m}^{14} \\ r_a^{-1} P_a^{21} & r_a^{-1} P_a^{22} & r_a^{-1} P_a^{23} & r_a^{-1} P_a^{24} \\ r_b^{-1} P_b^{31} & r_b^{-1} P_b^{32} & r_b^{-1} P_b^{33} & r_b^{-1} P_b^{34} \\ r_c^{-1} P_c^{41} & r_c^{-1} P_c^{42} & r_c^{-1} P_c^{43} & r_c^{-1} P_c^{44} \end{bmatrix} \begin{bmatrix} X_{a_m} \\ X_a \\ X_b \\ X_c \end{bmatrix}$$

Using the above, we can write

$$\dot{X}_{a_m} - \dot{X}_a = -\Big[(r_{a_m}^{-1}P_{a_m}^{11} - r_a^{-1}P_a^{21})X_{a_m} + (r_{a_m}^{-1}P_{a_m}^{12} - r_a^{-1}P_a^{22})X_a + (r_{a_m}^{-1}P_{a_m}^{13} - r_a^{-1}P_a^{23})X_b + (r_{a_m}^{-1}P_{a_m}^{14} - r_a^{-1}P_a^{24})X_c\Big]$$

Since $r_{a_m} = r_a$ we have

$$\dot{X}_{a_m} - \dot{X}_a = -r_{a_m}^{-1}\Big[(P_{a_m}^{11} - P_a^{21})X_{a_m} + (P_{a_m}^{12} - P_a^{22})X_a + (P_{a_m}^{13} - P_a^{23})X_b + (P_{a_m}^{14} - P_a^{24})X_c\Big]. \quad (68)$$

Denoting $\Gamma_2(t) = (S_{a_m}P_{a_m} + S_aP_a + S_bP_b + S_cP_c)$ we write (67) for P_{a_m} and P_a as

$$\dot{P}_{a_m} = -\widetilde{Q}_{a_m} + P_{a_m}\Gamma_2(t) \quad (69)$$

$$\dot{P}_a = -\widetilde{Q}_a + P_a\Gamma_2(t) \quad (70)$$

Again using the partitioning (28) and pre-multiplying the (69) with the matrix $\begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$ and pre-multiplying (70) with $\begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$ we obtain.

$$\begin{bmatrix} \dot{P}_{a_m}^{11} & \dot{P}_{a_m}^{12} & \dot{P}_{a_m}^{13} & \dot{P}_{a_m}^{14} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix} + \begin{bmatrix} P_{a_m}^{11} & P_{a_m}^{12} & P_{a_m}^{13} & P_{a_m}^{14} \end{bmatrix}\Gamma_2(t),$$

$$\begin{bmatrix} \dot{P}_a^{21} & \dot{P}_a^{22} & \dot{P}_a^{23} & \dot{P}_a^{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix} + \begin{bmatrix} P_a^{21} & P_a^{22} & P_a^{23} & P_a^{24} \end{bmatrix}\Gamma_2(t).$$

Taking the difference of the above two differential equations we obtain

$$\begin{bmatrix} \dot{P}_{a_m}^{11} - \dot{P}_a^{21} & \dot{P}_{a_m}^{12} - \dot{P}_a^{22} & \dot{P}_{a_m}^{13} - \dot{P}_a^{23} & \dot{P}_{a_m}^{14} - \dot{P}_a^{24} \end{bmatrix} = \begin{bmatrix} P_{a_m}^{11} - P_a^{21} & P_{a_m}^{12} - P_a^{22} & P_{a_m}^{13} - P_a^{23} & P_{a_m}^{14} - P_a^{24} \end{bmatrix}\Gamma_2(t),$$

with terminal conditions $P_{a_m}^{1j}(t_k+T) - P_a^{2j}(t_k+T) = 0$ for $j = 1,2,3,4$. This implies that $P_{a_m}^{1j}(t) - P_a^{2j}(t) = 0$ for all $t \in [t_k, t_k+T]$. Using this in (68) we obtain $\dot{X}_{a_m} - \dot{X}_a = 0$. This implies that the target a_m remains constant distance and orientation with target a in the rescue mode. ∎

Remark 12. In the rescue mode, all the targets maximize their distance with the attacker, and minimize their distance with the defender. So, these two opposing behaviors result in the distance between the targets a_m and $a \in \mathscr{A} \backslash \{a_m\}$ to remain constant. We let weights in the target's objectives (5) as $Q_{ab} = Q_{abT} = q_{ab}I$ and $Q_{ac} = Q_{acT} = q_{ac}I$, with $q_{ab} > 0$ and $q_{ac} > 0$. If $q_{ab} > q_{ac}$ then targets give more weightage on rendezvousing with the defender than evading the attacker, and vice-versa when the weights satisfy $q_{ab} < q_{ac}$.

The next result says that the switching policy defined by (25) results in the attacker locking on a target as soon the game switches to the interception mode.

Theorem 7. *Let Assumptions 1 and 2 hold true. Let t_k be the time instant when the game switches to the interception mode from the rescue mode according to the switching rule $\Psi(X(t))$ defined by (25). Let the minimum distance target at time t_k be*

$$a^* := \arg\min_{a\in\mathscr{A}} ||X_c(t_k) - X_a(t_k)||_2. \quad (71)$$

Then, the interception mode is invariant. Further, the attacker locks on to the target a^* for the remaining part of the game.

Proof. As the game enters the interception mode at t_k we have from (25) that $||X_c(t_k) - X_{a^*}(t_k)||_2 \leq \kappa\sigma_c$. Next, for the time duration $[t_k, t_{k+1})$ we know from Lemma 1 that $||X_c(t) - X_{a^*}(t)||_2$ is a strictly decreasing function of time for $t \in [t_k, t_{k+1})$. Moreover, from Theorem 5 we have that the target a^* remains to be the minimum distance target at t_{k+1} as well. In particular, we have that

$$||X_c(t_{k+1}) - X_{a^*}(t_{k+1})||_2 < ||X_c(t_k) - X_{a^*}(t_k)||_2 \leq \kappa\sigma_c,$$

implying that mode switching cannot happen at the time instant t_{k+1} and the game continues in the interception mode during the time period $[t_{k+1}, t_{k+2})$. Using the same arguments at next time instant t_{k+2} we infer that interception mode is invariant. Furthermore, as the target a^* given by (71) remains to be the minimum distance target at every time instant, the attacker locks on to a^* from t_k on wards. ∎

VI. Simulation Results

In this section, we illustrate the performance of switching strategies, developed in section IV, through numerical experiments. We consider a 5-player game consisting of three targets, one defender and one attacker. We analyze two scenarios. In Scenario-1, we verify the results developed in section V and analyze the effect of varying the parameters $Q_{ab} = Q_{abT} = q_{ab}I$, $Q_{ac} = Q_{acT} = q_{ac}I$ for $a \in \mathscr{A}$ and the planning horizon T. In Scenario-2, we analyze the effect of switching function parameter κ in (25), the degree of alertness of the defender, on the outcome of the game.

Scenario-1: Initially, the three targets a_1, a_2, and a_3 are located at $(0.866, 0.5)$, $(-0.866, 0.5)$, and $(0, -1)$ respectively. The defender b and the attacker c are located at $(-4, 2)$ and $(4, 4)$ respectively. The parameter values for the baseline case are taken as follows: $q_{ab} = 1$, $q_{ac} = 1$, for $a \in \{a_1, a_2, a_3\}$, $R_{a_1} = R_{a_2} = R_{a_3} = 400I$, $R_b = 300I, R_c = 150I$, and $Q_{ba} = Q_{baT} = Q_{ca} = Q_{caT} = I$ for $a \in \{a_1, a_2, a_3\}$, $Q_{bc} = Q_{bcT} = I$. As $r_a > r_b > r_c$, we have that the targets and defender penalize their control efforts more than the attacker. Other parameters are taken as follows: $T = 15$, $\delta = 0.02$, $\sigma_b = \sigma_c = 0.5$ and $\kappa = 5$. For the baseline case, Figure 3a illustrates the trajectories of the players, and Figure 3b illustrates the distances between the players. The defender starts in the rescue mode and switches to interception mode at $t_k = 0.86$, when the distance between the attacker c and its minimum distance target a_1 is less than or equal to $\kappa\sigma_c = 2.5$. The distances between a_1 and other targets a_2 and a_3 remain constant in the rescue mode verifying Theorem 6. From Figure 3c, the slope of the lines joining the attacker and the target a_1 is constant at $38.6752°$ in the interception mode. This verifies Theorem 3. Again, the slopes of the lines joining the target a_1 with a_2 and a_3 remain constant at $0°$ and $60°$ verifying Theorem 4. Next, the planning horizon T satisfies the condition (62) with $k = 0$, as $T = 15 \in (-0.9792, 30.4168)$. This implies, Assumption 2 holds true. From Figure 3b, in the interception mode (after $t > 0.86$), the distance between the attacker and the target a_1 decreases with time. Further, the distance between the targets a_1 with a_2 and a_3 increases with time. These observations verify Lemma 1 and Theorem 5. Further, the attacker locks on to the target a_1 after $t_k > 0.86$, thus verifying the prediction from Theorem 7. From Figure 3b the distance between the defender and the attacker equals the capture radius σ_b at time $t_k = 1.24$, implying that the defender intercepts the attacker. From Remark 12, when the parameter q_{ab} takes values greater than $q_{ac} = 1$, then the inter target distance decreases as the targets emphasize rendezvousing with the defender more than evading the attacker. Figure 3d illustrates this observation when q_{ab} is taken as 4.75, where the outcome of the game results in rescue of all the targets. When the parameter q_{ac} is set to $1.75 > q_{ab} = 1$, then the inter target distance increases as the targets now emphasize evading the attacker more than rendezvousing with the defender. Figure 3e illustrates this observation where the defender intercepts the attacker. Next, we analyze the effect of varying the planning horizon length T. In the baseline case, T satisfies the condition (62). Now, we set $T = 50$ so as to satisfy the other condition (63) with $k = 1$, that is, $T = 50 \in (30.4368, 61.8327)$. Figure 3f and 3g illustrates the trajectories of the players and distances between the players respectively. From Figure 3g, it can be seen that the distance between the target a_1 with a_2 and a_3 decreases with time in the interception mode (after $t_k > 0.68$). This observation again verifies Lemma 1.

Scenario-2: Initially, the three targets a_1, a_2 and a_3 are located at $(-.75, 0)$, $(0, 0)$, and $(.75, 0)$ respectively. The defender b and the attacker c are located at $(4, 4), (0, 6)$ respectively. The parameter values are taken as follows: $q_{ab} = 1$, $q_{ac} = 1$, for $a \in \{a_1, a_2, a_3\}$, $R_{a_1} = R_{a_2} = R_{a_3} = 400I$, $R_b = 300I$, $R_c = 250I$, and $Q_{ba} = Q_{baT} = Q_{ca} = Q_{caT} = I$ for $a \in \{a_1, a_2, a_3\}$, $Q_{bc} = Q_{bcT} = I$. Other parameters are taken as follows: $T = 15$, $\delta = 0.02$, $\sigma_b = \sigma_c = 0.75$. First we set the parameter $\kappa = 1$ and the game starts in rescue mode. Figure 3h illustrates the trajectories of the players. Figure 3i illustrates the distance between the attacker c and the targets a_1 and a_2. At the time instant $t_k = 0.58$, the attacker updates its minimum distance target from a_2 to a_1. The game terminates at $t_k = 2.04$ with attacker capturing the target a_1. Next, when the parameter κ is increased to 3, indicating a highly alert defender, it can be observed from Figure 3j that the defender switches from rescue mode to interception mode at $t_k = 1.1$ and eventually intercepts the attacker at $t = 1.26$. As the parameter κ only influences the defender's ability to switch the operational behavior, the behavior of the player before the mode switch at time instant $t_k = 1.1$ is identical to the situation where $\kappa = 1$. This implies that the attacker updates the minimum distance target from a_2 to a_1 at time instant $t_k = 0.58$ for this case as well. It can be observed that in the interception mode the lines joining the minimum distance target a_1 with the other targets remain parallel verifying Theorem 4. Further, the inter target distance remains constant in the rescue mode verifying Theorem 6.

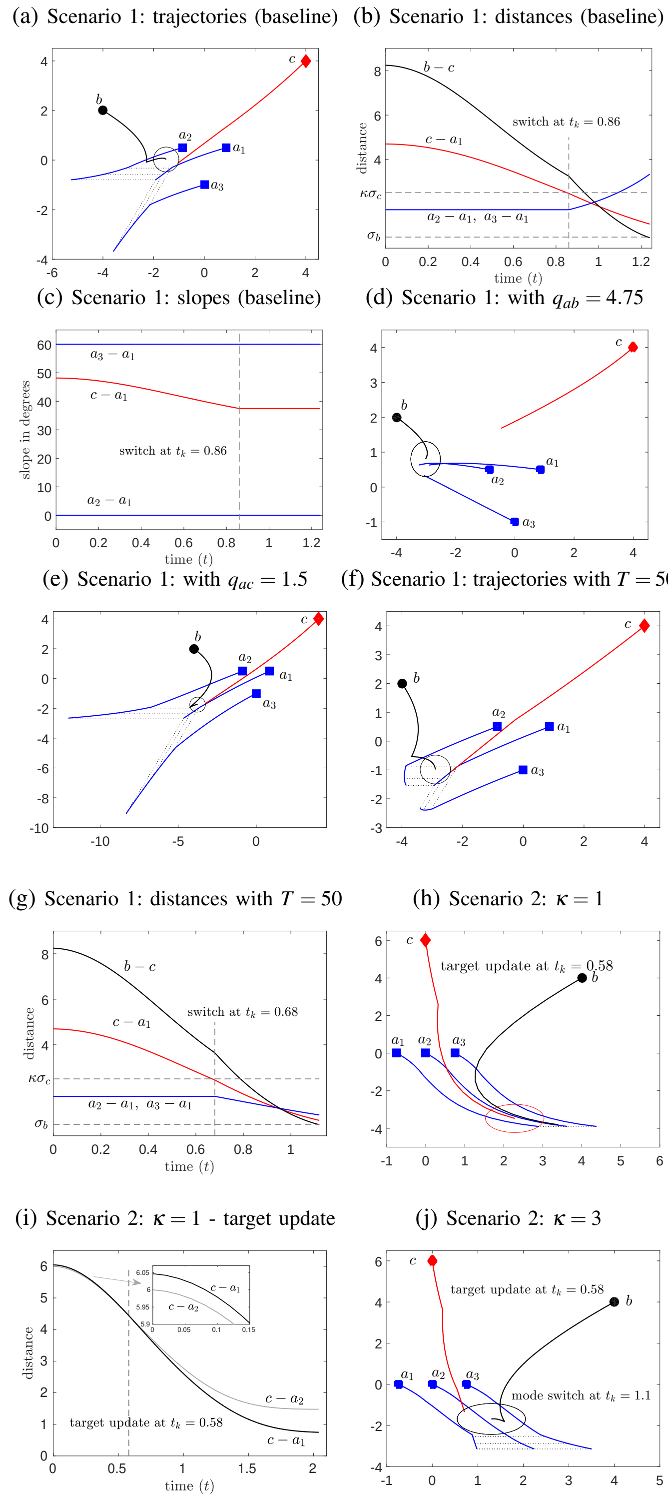

(a) Scenario 1: trajectories (baseline) (b) Scenario 1: distances (baseline) (c) Scenario 1: slopes (baseline) (d) Scenario 1: with $q_{ab} = 4.75$ (e) Scenario 1: with $q_{ac} = 1.5$ (f) Scenario 1: trajectories with $T = 50$ (g) Scenario 1: distances with $T = 50$ (h) Scenario 2: $\kappa = 1$ (i) Scenario 2: $\kappa = 1$ - target update (j) Scenario 2: $\kappa = 3$

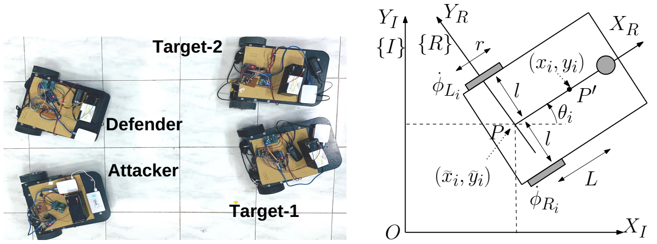
Fig. 3: In panels (h) and (j) the x-axis is stretched for better visibility of the interaction between the players. In panels (a), (e) (f) and (j) the dotted lines illustrate the lines joining the minimum distance target and the other targets in the interception mode. Black and red circles represent the capture zone of the defender and attacker respectively.

VII. Experimental Study

In this section, we illustrate dynamic game model and the implementation of the Algorithm 1 through experiments with players taken as differential drive mobile robots (DDMR). We present the robot model, discuss the experimental setup and illustrate some of results obtained in section V.

A. The robot model and feedback linearization

A differential drive mobile robot (DDMR) with two motorized fixed standard wheels and one unpowered omni-



(a) (b)

Fig. 4: Panel (a) illustrates the 4 DDMRs used in the experiments. Panel (b) illustrates the DDMR model showing inertial frame of reference with axes (X_I, Y_I) and robot's frame of reference with axes (X_R, Y_R)

directional castor wheel is shown in Figure 4b. Here, $\{I\}$ denotes the inertial frame of reference with origin O and basis (X_I, Y_I). $\{R\}$ corresponds to the local frame of reference having position P and basis (X_R, Y_R). The position of the robot in the inertial frame of reference is given by $(\tilde{x}_i, \tilde{y}_i)$ while θ_i, $i = \{\mathscr{A}, b, c\}$ corresponds to the angular difference between frames. The dynamics of robot i is given by; see [33], [34],

$$\dot{\tilde{x}}_i = \left(\frac{r\dot{\phi}_{Ri} + r\dot{\phi}_{Li}}{2}\right)\cos\theta_i,\ \dot{\tilde{y}}_i = \left(\frac{r\dot{\phi}_{Ri} + r\dot{\phi}_{Li}}{2}\right)\sin\theta_i, \tag{72a}$$

$$\dot{\theta}_i = \frac{r\dot{\phi}_{Ri} - r\dot{\phi}_{Li}}{2l}, \tag{72b}$$

where $2l$ is the distance between the wheels and r is the diameter of the wheel. The angular velocities of the right wheel $(\dot{\phi}_{Ri})$ and left wheel $(\dot{\phi}_{Li})$ are the control inputs with $(\tilde{x}_i, \tilde{y}_i, \theta_i)$ as the pose of the robot in robot frame $\{R\}$ at time t. Let v_i and ω_i be the translational and angular velocities of the robot respectively. Then we have $v_i = \frac{r\dot{\phi}_{R_i} + r\dot{\phi}_{L_i}}{2}$, $\omega_i = \frac{r\dot{\phi}_{R_i} - r\dot{\phi}_{L_i}}{2l}$. The DDMR dynamics can be rewritten as the following unicycle dynamics [35, Chapter 2]

$$\dot{x}_i = v_i \cos\theta_i,\ \dot{y}_i = v_i \sin\theta_i,\ \dot{\theta}_i = \omega_i. \tag{73}$$

However, for implementation purposes, the actual control inputs $\dot{\phi}_{Ri}$, $\dot{\phi}_{Li}$ are obtained from (72) and (73) as

$$\dot{\phi}_{R_i} = \frac{1}{r}(v_i + l\omega_i),\ \dot{\phi}_{L_i} = \frac{1}{r}(v_i - l\omega_i). \tag{74}$$

The robot dynamics given by equation (72) is non-linear, and a dynamic game formulation is difficult to solve in general. We therefore use feedback linearization [35, Chapter 2] and then apply our LQDG framework.

Let P be the origin of the robot in robot frame and P' be the center of mass at a distance L from the origin P as shown in Figure 4b. For robot $i \in \mathscr{P}$, the coordinates of P' are

$$x_i = \tilde{x}_i + L\cos\theta_i,\ y_i = \tilde{y}_i + L\sin\theta_i.$$

Upon differentiating the above and using (73) we get,

$$\dot{x}_i = v_i \cos\theta_i - L\omega_i \sin\theta_i,\ \dot{y}_i = v_i \sin\theta_i + L\omega_i \cos\theta_i. \tag{75}$$

We define the following state feedback laws

$$v_i = \cos(\theta_i)u_{1i} + \sin(\theta_i)u_{2i} \tag{76a}$$

$$\omega_i = \frac{1}{L}\left(-\sin(\theta_i)u_{1i} + \cos(\theta_i)u_{2i}\right), \tag{76b}$$

and then using (76a) in (75) we get,

$$\dot{x}_i = u_{1i},\ \dot{y}_i = u_{2i}. \tag{77}$$

The LQDG formulation considers the point P' and provides the Nash equilibrium controls (u_{1i}, u_{2i}) for robot i. For implementation, the actual controls $(\dot{\phi}_{Ri}, \dot{\phi}_{Li})$ are obtained using (76a) and (76b) in (74) as

$$\dot{\phi}_{Ri} = \frac{\cos(\theta_i)}{r}\left(u_{1i} + \frac{u_{2i}l}{L}\right) + \frac{\sin(\theta_i)}{r}\left(u_{2i} - \frac{u_{1i}l}{L}\right) \tag{78a}$$

$$\dot{\phi}_{Li} = \frac{\cos(\theta_i)}{r}\left(u_{1i} - \frac{u_{2i}l}{L}\right) + \frac{\sin(\theta_i)}{r}\left(u_{2i} + \frac{u_{1i}l}{L}\right). \tag{78b}$$

From the above equations it is evident that the feedback linearization parameter L must be chosen carefully.

B. Experimental setup and implementation details

The experiments employ four differential drive mobile robots that serve as an attacker, a defender and two targets (see Figure 4a). The distance between the wheels of a robot is $2l = 0.36m$ and the diameter of each wheel is $r = 0.13m$. Each of the robots has access to their initial location information. As the game progresses, each robot tracks and determines its local position and orientation with the help of Autonics E40H12 rotary encoders mounted on its two wheels. This state information is made available to the remaining players (and vice versa) over a wireless communication network. The implementation details of Algorithm 1 are given as follows. At a given time instant t_k, the computation of open-loop Nash equilibrium strategies over the planning horizon $[t_k, t_k + T]$ are performed using a Raspberry Pi 3 B+ board installed on each robot. Next, an on-board Arduino UNO is employed to enforce the control inputs on physical robots for the duration $[t_k, t_{k+1})$. Finally, a coordination protocol is adopted to synchronize the time instants t_k for the execution of control inputs (78) on each robot. The feedback linearization parameter L relates the

(a) Angular velocities (b) Orientation

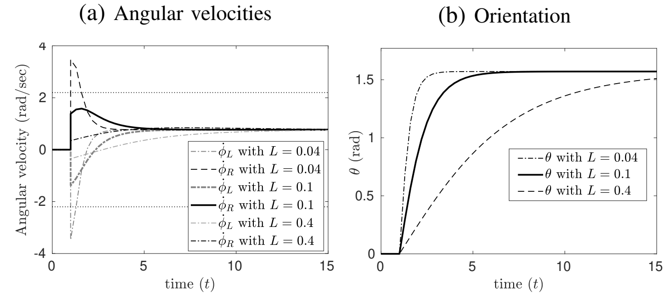


Fig. 5: DDMR controls and orientation

control inputs obtained from Algorithm 1 to the respective wheel velocities of the robots in the experimental setup. While a small value of L is desirable, it results in higher wheel velocities. Figure 5a illustrates the wheel angular velocities for three values of L for a step input $u_1 = 0$ and $u_2 = 0.1m/s$ at $1s$. Here, these choice of inputs result in the maximum possible/worst-case rotation of the robot, which is $90°$. The dotted horizontal lines indicate the maximum achievable angular velocity of $21rpm (= 2.2rad/s)$ for the DC motors mounted on the robot wheels. When $L = 0.04m$, the maximum wheel angular velocities shoot up to a maximum of $3.46 rad/s$ which are impractical during implementation. Figure 5b illustrates the time taken by the robot to reach the desired rotation of 90°. It can be observed that a higher values of L the robot takes a longer time to reach the desired orientation. We thus adopt an intermediate value of $L = 0.1m$ that requires a maximum speed of $1.58 rad/s$ (see Figure 5a) which is well within the achievable speed of the DC motors ($2.2 rad/s$). Finally, the moving horizon time instant duration δ is taken as $0.5s$ to accommodate the time spent in inter robot communication and enforcement of determined wheel velocities on physical robots.

C. Experiment-1 (with one target)

In this experiment, we consider one target to illustrate the implementation of Algorithm 1. The initial P (P')-coordinates of the target, defender and the attacker are given by $(0.4,3,0°)(0.5,3,0°)$, $(3.4,3,180°)(3.3,3,180°)$ and $(1.8,4.3,270°)(1.8,4.2,270°)$ respectively. The third coordinate indicates the orientation of the robots with respect to positive x-axis. The parameters are set as $\delta = 0.5s$, $T = 45s$, $R_a = 380I$, $R_b = 350I$, $R_c = 300I$, $Q_{abT} = Q_{acT} = Q_{ab} = Q_{ac} = Q_{baT} = Q_{ba} = Q_{caT} = Q_{ca} = I$ and $Q_{bc} = Q_{bcT} = 5I$. The capture radii of the defender and attacker are set as $\sigma_b = \sigma_c = 0.5m$ with $\kappa = 3.2$. Since there is only one target the attacker always pursues this target. As the initial distance between the attacker and target is greater than $\kappa\sigma_c$ the game starts in the rescue mode. This distance equals $\kappa\sigma_c$ at 8 seconds and the game switches to interception mode; see Figure 6b. The game terminates with the attacker intercepted by the defender at time instant $t = 18s$, that is, when the attacker lies within the capture radius of the defender; see Figure 6a. Figure 6c illustrates the slope of the line joining the attacker and the target for the duration $[8s\ 18s]$ in the interception mode. We observe that the mean slope of this line is 43.4610° with standard deviation 0.8519°, implying that the slope is almost constant thus verifying Theorem 3. Video recording of the experiment is available at the link https://youtu.be/lK5AHigv_Rc.

D. Experiment-2 (with two targets)

In this experiment we consider two targets. The initial $P(P')$-coordinates of the targets (labeled as a_1 and a_2), defender and attacker are taken as $(3.5,2.5,270°)$ $(3.5,2.4,270°)$, $(6,2.5,180°)(5.9,2.5,180°)$, $(5.5,5.5,180°)(5.4,5.5,180°)$ and $(3.3,7,270°)(3.3,6.9,270°)$ respectively. The initialization parameters are set as $\delta = 0.5s$, $T = 45s$, $R_{a_1} = R_{a_2} = 480I$, $R_b = 350I$, $R_c = 280I$, $Q_{a_1b} = Q_{a_2b} = Q_{a_1bT} = Q_{a_2bT} = 2I$, $Q_{bc} = Q_{bcT} = 3I$ with other matrices taken as I. The capture radii of the defender and the attacker are taken as $\sigma_b = \sigma_c = 0.5$ with the switching parameter set as $\kappa = 4$. Based on switching condition, initially, the defender attempts to rescue both the target robots while the attacker pursues its closest target a_1. In our experiment, the target a_1 remains to be the minimum distance target for the attacker throughout the game. The game continues in the rescue mode for $25s$ when a_1 is at a distance of $\kappa\sigma_c = 2$

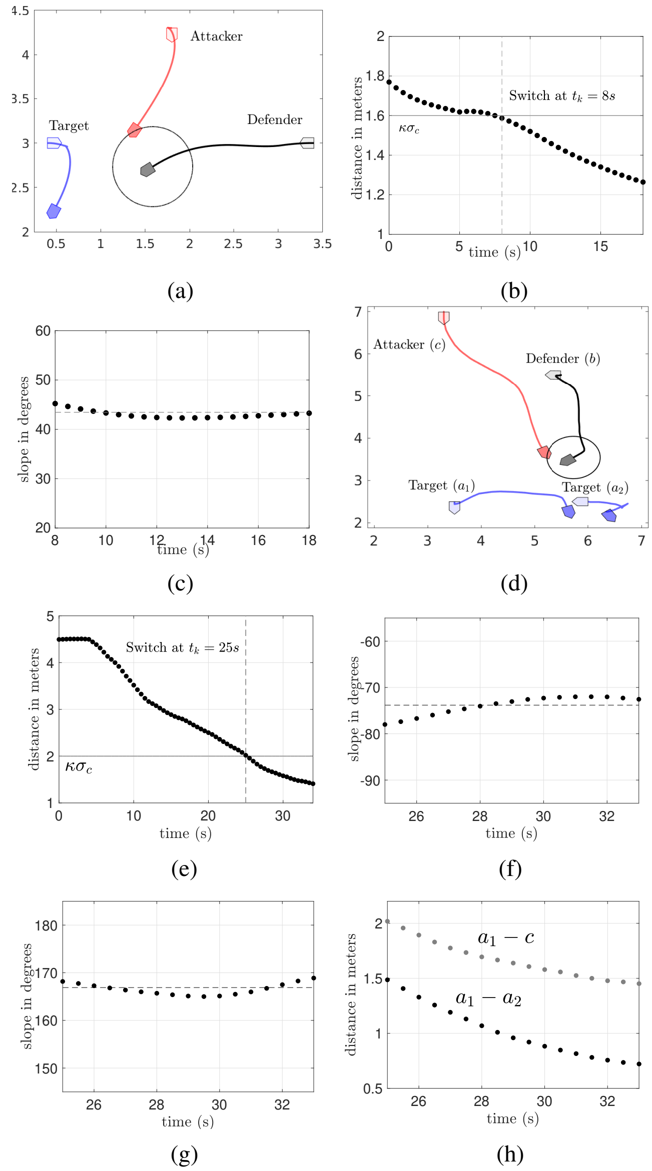


Fig. 6: *Experiment -1*: Panel (a) illustrates the P-trajectories of the attacker, defender and the target. Panel (b) depicts the distance between the target and the attacker. Panel (c) depicts the slope of the line joining the target and the attacker in the interception mode.

Experiment -2: Panel (d) illustrates the P-trajectories of the attacker, defender and the targets. Panel (e) illustrates the distance between the minimum distance target (a_1) and the attacker. Panel (f) illustrates the slope of the line joining the target (a_1) and the attacker in the interception mode. Panel (g) illustrates the slope of the line joining the targets (a_1 and a_2) in the interception mode. Panel (h) illustrates the distances between target a_1 with the attacker c and the target a_2

meters from the attacker; see Figure 6e. During the interval $[25s,\ 33s]$, that is, during the interception mode, Figure 6f illustrates the slope of the line joining between a_1 and c. We observe that the mean value of the slope is $-75.2123°$ with standard deviation 1.2968°, implying that the slope is almost constant thus verifying Theorem 3. Figure 6g illustrates the slope of the line joining the targets a_1 and a_2. We observe that the mean value of the slope is 166.8873° with standard deviation 1.5289°, implying that the slope is almost constant verifying Theorem 4. We notice that the parameters satisfy

$\frac{r_a - r_c}{r_a r_c} = 0.015 \in (0,1)$, and the planning horizon T satisfies condition (63) with $k = 1$, as $T = 45 \in (33.9151, 67.8295)$. Figure 6h illustrates the distance between the target a_1 and the attacker decreases with time and the inter target distance also decreases with time. This observation verifies Lemma 1. Video recording of the experiment is available at the link https://youtu.be/3cYzLHd8eZ4.

VIII. Conclusions

In this paper, we have analyzed a multiple Active Target-Attacker-Defender differential game where the defender adaptively switches operating in rescue and interception modes, and the attacker pursues the closest target during the course of the game. We model the interactions within each mode as LQDG and derive open-loop Nash equilibrium strategies of the players. Then, to enable switching we use receding horizon approach to obtain switching strategies for the players. Under few assumptions on the problem parameters, we characterized the geometrical properties of the trajectories of the players. Further, we also derived conditions under which the attacker locks on to a target. We illustrated our results with numerical simulations. Further, we demonstrated the performance of switching strategies using differential drive mobile robots and verified our results.

The ATAD model studied in our paper can be easily adapted to incorporate multiple defenders and attackers. For future work, we plan to investigate different cooperation situations between the targets and the defender, various criteria for switching and terminating the game, and the presence of obstacles.

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