Study of Multiple Target Defense Differential Games Using Receding Horizon-Based Switching Strategies

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***Abstract*— In this article, we study a variation of the active target-attacker-defender (ATAD) differential game involving multiple targets, an attacker, and a defender. Our model allows for 1) a capability of the defender to switch roles from *rescuer* (rendezvous with all the targets) to *interceptor* (intercepts the attacker) and vice versa and 2) the attacker to continuously pursue the closest target (which can change during the course of the game). We assume that the mode of the defender (rescue or interception) defines the mode of the game itself. Using the framework of Games of a Degree, we first analyze the game within each mode. More specifically, the objectives of the players are taken as a combination of weighted Euclidean distances and penalties on their control efforts. We model the interaction of the players within each mode as a linear quadratic differential game (LQDG) and obtain the open-loop Nash equilibrium strategies. We then use the receding horizon approach to enable switching between the modes to obtain switching strategies for the players. By partitioning the matrices associated with the Riccati differential equations we obtain geometric characterization of the trajectories of the players. Furthermore, under mild restrictions on the problem parameters and for a particular choice of the defender's switching function we show that interception mode is invariant. We illustrate our results with numerical simulations. Experimental results involving multiple autonomous differential drive mobile robots are presented.**

***Index Terms*— Autonomous multiagent systems, Nash equilibrium, pursuit evasion differential games, receding horizon approach, switching strategies, target-attacker-defender (TAD) differential games.**

I. Introduction

THE study of autonomous multiagent interactions has received considerable interest in recent years. This is mainly due to their applicability in modeling complex strategic phenomena arising in areas such as surveillance, rescue missions, combat operations, navigation, and analysis of biological behaviors. This article is concerned with analyzing strategic situations observed in the engineering applications such as a defense system protecting critical infrastructures (e.g., air crafts, naval ships) against attacks from incoming missiles, interceptor defending an asset against intrusions, and biological behavior such as mothers protecting young from potential attacks by the predators.

A common feature in the above strategic situations is the presence of multiple agents which are at conflict or cooperation that evolves over time. These situations are usually analyzed using the mathematical framework of pursuit-evasion games with three players—Target, Attacker and Defender–and referred to as a target-attacker-defender (TAD) game. Here, the goal of the attacker is to *capture* the target which tries to evade the attacker, and the goal of the defender is to *intercept* the attacker before the attacker captures the target. In a TAD game the target is assumed to be nonreactive (stationary or moves on a prescribed trajectory), and when the target is maneuverable then the interaction is referred to as an active TAD (ATAD) game; see references in Section I-B. Rescue type of interactions in the scenarios mentioned above can be modeled using the framework of a Prey-Protector-Predator (P3) game which was introduced in [1]. Here, the goal of the protector is to rendezvous with the prey in order to *rescue* the prey before it is captured by the predator.

A. Contributions

A majority of the existing literature on (A)TAD and P3 games consider three-player engagements. In the real-world applications, for example, combat operations, rescue missions, protection of young and coordinated hunting in the animal world often involve multiple ($n \geq 3$) players. Further, in almost all the existing works, the interactions between the players are fixed throughout the duration of the game. In the real-world scenarios, these interactions often change during the course of the engagement. For example, the defender may find it economical to rescue the targets instead of intercepting the attacker from the onset of the game. Only at an opportune moment, when the threat level reaches a certain threshold, the defender may want to switch to intercepting the attacker. Further, due to the presence of multiple targets, the attacker may want to dynamically update the target it wants to capture as the game evolves in time.

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The contribution of this article is to introduce a framework for studying dynamically evolving multiagent interactions of ATAD type. In particular, the novelty of our work lies in the consideration of the presence of multiple targets, and a flexible (and powerful) capability of the defender to autonomously switch roles from being a rescuer to interceptor and vice versa. Introducing these two features naturally leads to challenging questions such as 1) can the trajectories of the attacker and the target it pursues be geometrically characterized? 2) how do the trajectories of targets evolve when the defender acts as a rescuer and as an interceptor? and 3) under what conditions will the attacker lock on to a target and pursues it forever? To address these questions we consider a model where players engage in two types of interactions, also called as *modes*, based on the role of the defender. In the *rescue* mode, the defender attempts rendezvous with the targets, whereas in the *interception* mode it tries to intercept the attacker. The attacker tries to capture a closest target and all targets try to evade the attacker. The defender is capable of autonomously switching the roles based on the state of the game. Our work distinguishes from the existing literature where the interactions are fixed for the entire duration of the engagement. To achieve our objective, first we fix the interactions of the players in one of the modes alone. Using the Games of a Degree approach, the interaction among the players in a mode is formulated as a finite horizon nonzero sum linear quadratic differential game (LQDG); similar approach was followed in the works [2] and [3], and the open-loop Nash equilibrium control strategies of the players are computed. To facilitate switching between the modes, we adopt the receding horizon approach to obtain switching strategies of the players. The main results of our article are summarized as follows.

1) In the interception mode, we show in Theorem 3 that the attacker and its closest target move on a straight line joining their initial locations.
2) In the interception mode, we show in Theorem 4 when the targets are identical, then the closest target (to the attacker) and other targets undergo parallel evolution.
3) In interception mode, we show in Lemma 1 that the distance between the closest target (to the attacker) and other targets either increase or decrease depending upon the bounds placed on the planning horizon length. Using this result, and with a particular form of defender's switching function we show in Theorem 6 that the interception mode is invariant, and the attacker locks on to a target for the remaining duration of the game, thereby demonstrating the convergence of the mode induced by the switching policy.
4) In the rescue mode, we show in Theorem 7 that the distance between the closest target (to the attacker) and the other targets remains constant.

The article is organized as follows. In Section II, we present dynamics of the players and their interactions. In Section III, we solve the LQDG assuming that the mode of the game is restricted to either rescue or interception alone and derive the open-loop Nash equilibrium strategies of the players. In Section IV, we augment the open-loop Nash equilibrium strategies with receding horizon approach to enable switching and provide an algorithm for computing the switching strategies of players. In Section V, we provide results related to the behavior of the players. In Section VI, we illustrate our results with numerical simulations. Toward a practical realization of our study, in Section VII, we illustrate our results taking differential drive mobile robots (DDMRs) as players. Finally, Section VIII provides concluding remarks and a summary of future research.

B. Overview of the Related Literature

TAD-type interactions were studied in [4] and [5] in the context of defending ships from an incoming torpedo using counter-weapons. In [6], a two-player differential game of target defense is studied, where the objective of one player is to drive the state of the system to reach the target whereas the other player requires the state to avoid the target. A TAD type interaction referred to as the lady, the bandits, and the body-guards was proposed in [7]. In [8], the authors study an ATAD terminal game and propose attacker strategies for evading the defender while continuing to pursue the target. In [2], the authors study the problem of defending an asset. Here, the interactions are modeled as a LQDG. In [9], a guidance law for defending a nonmaneuverable aircraft is proposed, and a real-time target guarding problem was studied in [10]. In [11] and [12], line-of-sight and other guidance laws are presented for defending aerial targets. In [13], [14], [15], and [16], the authors consider various cooperation scenarios between the aircraft (target) and the defensive missile (defender) against the incoming homing missile (attacker). ATAD type interactions can be found in applications such as territory or boundary guarding; see [17] and [18]. P3 type of interactions were investigated by Oyler *et al.* [1]. In [19], the authors study a P3 type interaction of vision-guided predator with multiple protectors and prey robots.

In a series of works [20]–[23], the authors consider an ATAD game where a homing missile (attacker) tries to pursue an aircraft (target), and a defender missile aims at intercepting the attacker in order to protect the target. In [24], the ATAD interaction is posed as a zero-sum differential game between the defender-target team and the attacker. In [25], the same game is studied and the authors construct barrier surfaces and characterize the escape and capture regions for the target. In [26], the authors study a TAD interaction as an LQDG and provides closed-form solutions for players' strategies through the analysis of the associated coupled Riccati differential equations. In [27], the authors study an LQDG with ATAD interactions where the target has a predefined goal besides evading the attacker.

Related to the literature on role switching in ATAD games, in the recent work [28], the authors study a three-player ATAD game where the survivability of the attacker is of importance. In this model, the attacker is allowed to switch from pursuing the target to evading the defender at an opportune moment. In [29], the defender's strategies force the attacker to retreat instead of engaging the target. In [30], the authors study the possibility of role switch as well as the cooperation between the target and defender. In almost all these works, the role of the defender is to intercept the attacker and do not consider the role switch by the defender.

A preliminary conference version of this article appeared in [31] where a three-player game is studied and does not consider the presence of multiple targets. This article goes much beyond the work [31], both in content and scope, by providing proofs for the analytical characterizations of trajectories, and illustrations with experiments.

Notation: Throughout this article, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. The symbol $\otimes$ denotes the Kronecker product. The transpose of a vector or matrix E is denoted by E'. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $||x||_2 = (x'x)^{1/2}$. For any $x \in \mathbb{R}^n$ and $S \in \mathbb{R}^{n\times n}$, we denote the quadratic term $x'Sx$ by $||x||_S^2$. $\text{diag}\{e_1, e_2, \dots, e_n\}$ denotes the block diagonal matrix obtained by taking the scalars or matrices $e_1, e_2, \dots, e_n$ as diagonal elements in this sequence. $\mathbf{I}_n$ denotes the $n \times n$ matrix. $\mathbf{0}_n$ and $\mathbf{0}_{n\times m}$ denote the $n \times n$ and $n \times m$ matrices of all zeros, respectively.

II. Multiple ATAD Differential Game

In this section, we describe the interactions and dynamics of the players and provide the dynamic game methodology for modeling players' interactions.

A. Dynamics of the Players

We consider a team of n active targets which are pursued by one attacker. We assume the availability of one defender whose task is to either save or *rescue* all the targets or to *intercept* the attacker. We denote the set of n targets by $\mathscr{T} := \{\tau_1, \tau_2, \dots, \tau_n\}$, the defender by d and the attacker by a. The set of players is denoted by $\mathscr{P} := \mathscr{T} \cup \{a, d\}$. We assume that the players interact in a 2-D plane. The dynamics of each player is governed by the following single integrator dynamics:

$$\begin{bmatrix}\dot{x}_i(t)\\ \dot{y}_i(t)\end{bmatrix} = \begin{bmatrix}u_{1i}(t)\\ u_{2i}(t)\end{bmatrix}, \quad \begin{bmatrix}x_i(0)\\ y_i(0)\end{bmatrix} = \begin{bmatrix}x_{i0}\\ y_{i0}\end{bmatrix} \tag{1}$$

where $(x_i(t), y_i(t)) \in \mathbb{R}^2$ is the position vector of the player $i \in \mathscr{P}$ at time t, $(u_{1i}(t), u_{2i}(t)) \in \mathbb{R}^2$ represents the control input of player i at time t, and $(x_{i0}, y_{i0}) \in \mathbb{R}^2$ represents the initial position vector of player i. We denote the state and control vector of player $i \in \mathscr{P}$ as

$$X_i(t) = \begin{bmatrix}x_i(t)\\ y_i(t)\end{bmatrix}, \quad u_i(t) = \begin{bmatrix}u_{1i}(t)\\ u_{2i}(t)\end{bmatrix}. \tag{2}$$

By denoting $X(t) = [X'_{\tau_1}(t)\ X'_{\tau_2}(t)\ \cdots\ X'_{\tau_n}(t)\ X'_d(t)\ X'_a(t)]'$, the dynamic interaction environment of the players can be written compactly as follows:

$$\dot{X}(t) = \left(\sum_{j=1}^{n} B_{\tau_j} u_{\tau_j}(t)\right) + B_d u_d(t) + B_a u_a(t) \tag{3}$$

where $B_{\tau_j} = [d_1\ \ d_2 \cdots\ d_n\ \ 0\ \ 0]' \otimes \mathbf{I}_2$ with $d_j = 1$, $d_l = 0,\ \forall\ l \neq j$, $B_d = [0\ \ 0\ \ \cdots\ \ 0\ \ 1\ \ 0]' \otimes \mathbf{I}_2$, $B_a = [0\ \ 0\ \ \cdots\ \ 0\ \ 0\ \ 1]' \otimes \mathbf{I}_2$.

Remark 1: The methodology and results presented in the article can be extended easily to an n-dimensional setting.

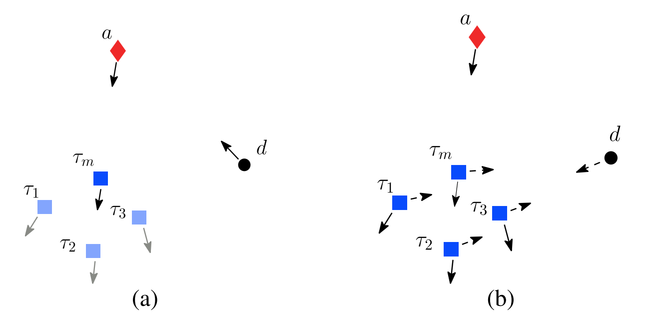


Fig. 1. Interaction of players in both the modes. τ_m is a target which is at a minimum distance to the attacker a. (a) Interception mode. (b) Rescue mode.

B. Players' Interactions as a Differential Game

In our article, the interactions between the players are described as follows.

- **I1.** The attacker always tries to capture a target which is at the closest distance to it.
- **I2.** The defender can operate in two modes namely rescue or interception modes. In the interception mode, the defender tries to intercept the attacker, whereas in the rescue mode the defender tries to rendezvous with all the targets (in order to save them). The defender is capable of switching the operational modes autonomously depending upon the state of the game.
- **I3.** The targets try to evade the attacker–individually, without forming a team–in the interception mode. In the rescue mode, besides evading the attacker they also attempt to rendezvous with the defender.

Fig. 1 illustrates the interaction among the players in both the operational modes. Our work distinguishes from the existing literature due to the presence of the following three features in the interactions (I1–I3).

- **F1.** The attacker always pursues a target that is at the minimum distance to it. In our setting, the target which is at a minimum distance to the attacker keeps changing with time.
- **F2.** As the defender can autonomously switch operating in rescue mode to interception mode and vice-versa, the players with whom the defender is in direct conflict/cooperation also changes with time.
- **F3.** The targets are in conflict with the attacker both in rescue and interception modes, and they are in cooperation with the defender in rescue mode.

C. Termination Criterion

The outcomes of the interactions are described as follows. We denote the positive real numbers σ_d and σ_a as the capture radii of defender and attacker, respectively. In the rescue mode, the interactions terminate when the defender rescues all the targets, that is, whenever $||X_\tau(t) - X_d(t)||_2 \leq \sigma_d$ for all $\tau \in \mathscr{T}$, or when attacker captures at least one target, that is, whenever $||X_a(t) - X_\tau(t)||_2 \leq \sigma_a$ for at least one $\tau \in \mathscr{T}$. Similarly, in the interception mode, the interactions terminate when the defender intercepts the attacker, that is, whenever

$||X_d(t) - X_a(t)||_2 \leq \sigma_d$ or when the attacker captures a target, that is, whenever $||X_a(t) - X_\tau(t)||_2 \leq \sigma_a$ for at least one $\tau \in \mathscr{T}$.

Remark 2: In the ATAD differential games literature, usually two approaches are followed for analyzing the interactions of the type (I1–I3), the *Game of Kind* (GoK) and the *Game of Degree* (GoD). The former approach determines the possible outcomes of the game from the given initial conditions. In the latter approach performance metrics–for instance, Euclidean distance between the players–are used toward quantifying the outcome of the game. Furthermore, it is possible to embed a GoK within the framework of a GoD; see [32]. In this article, we use GoD methodology for analyzing the interactions of the type (I1–I3); see also Section VI.

III. Analysis in Rescue or Interception Mode

In this section, we study the game where the interactions of the players are fixed in the rescue or interaction mode alone. Let the target which is at a minimum distance to the attacker at the initial time $t_0 = 0$ be denoted by τ_m. Then τ_m satisfies

$$\tau_m := \arg\min_{\tau \in \mathscr{T}} ||X_a(0) - X_\tau(0)||_2. \tag{4}$$

First, we consider the setting where the interaction pattern of the players is fixed to be either in the rescue or interception mode for a time duration $T > 0$. We assume that the attacker pursues the target τ_m throughout the time duration $[0, T]$. Furthermore, we also assume that the defender operates in one of these modes for the duration $[0, T]$. In the rescue mode, the defender minimizes the sum of weighted Euclidean distances to all the targets. In this mode, the targets in $\mathscr{T}$ minimize and maximize their weighted Euclidean distance with the defender and the attacker, respectively. In the interception mode, the defender minimizes its weighted Euclidean distance with the attacker while the targets in $\mathscr{T}$ maximize their weighted Euclidean distances with the attacker. In our work we allow for variable speeds of the players. Further, all the players simultaneously minimize the energy expenditure i.e., the control effort to be consumed in (both) the modes. The performance metric to be minimized by player $i \in \mathscr{P}$ is then given by

$$J_i\big(u_{\tau_1}(.), \ldots, u_{\tau_n}(.), u_d(.), u_a(.), t_0, T\big) := G_i(t_0 + T) + \int_{t_0}^{t_0+T} L_i(t)dt \tag{5}$$

where $G(T) = G_i(t_0 + T)$ (as $t_0 = 0$) and $L_i(t)$ denote the terminal and running costs of player $i \in \mathscr{P}$. These costs for players $\{\tau_j \in \mathscr{T}, d, a\}$ are given by

$$G_{\tau_j}(T) = \frac{\alpha_R}{2}\big\|X_{\tau_j}(T) - X_d(T)\big\|^2_{Q_{\tau_j dT}} - \frac{1}{2}\big\|X_{\tau_j}(T) - X_a(T)\big\|^2_{Q_{\tau_j aT}} = \frac{1}{2}\|X(T)\|^2_{\widetilde{Q}_{\tau_j T}} \tag{6}$$

$$\begin{aligned} L_{\tau_j}(t) &= \frac{1}{2}\big\|u_{\tau_j}(t)\big\|^2_{R_{\tau_j}} + \frac{\alpha_R}{2}\big\|X_{\tau_j}(t) - X_d(t)\big\|^2_{Q_{\tau_j d}} - \frac{1}{2}\big\|X_{\tau_j}(t) - X_a(t)\big\|^2_{Q_{\tau_j a}} \\ &= \frac{1}{2}\big\|u_{\tau_j}(t)\big\|^2_{R_{\tau_j}} + \frac{1}{2}\big\|X(t)\big\|^2_{\widetilde{Q}_{\tau_j}} \end{aligned} \tag{7}$$

$$G_d(T) = \frac{\alpha_R}{2}\sum_{j=1}^{n}\|X_{\tau_j}(T) - X_d(T)\|^2_{Q_{d\tau_j T}} + \frac{\alpha_I}{2}\|X_d(T) - X_a(T)\|^2_{Q_{daT}} = \frac{1}{2}\|X(T)\|^2_{\widetilde{Q}_{dT}} \tag{8}$$

$$\begin{aligned} L_d(t) &= \frac{1}{2}\|u_d(t)\|^2_{R_d} + \frac{\alpha_R}{2}\sum_{j=1}^{n}\|X_{\tau_j}(t) - X_d(t)\|^2_{Q_{d\tau_j}} + \frac{\alpha_I}{2}\|X_d(t) - X_a(t)\|^2_{Q_{da}} \\ &= \frac{1}{2}\|u_d(t)\|^2_{R_d} + \frac{1}{2}\|X(t)\|^2_{\widetilde{Q}_d} \end{aligned} \tag{9}$$

$$G_a(T) = \frac{1}{2}\big\|X_{\tau_m}(T) - X_a(T)\big\|^2_{Q_{a\tau_m T}} = \frac{1}{2}\|X(T)\|^2_{\widetilde{Q}_{aT}} \tag{10}$$

$$\begin{aligned} L_a(t) &= \frac{1}{2}\|u_a(t)\|^2_{R_a} + \frac{1}{2}\big\|X_{\tau_m}(t) - X_a(t)\big\|^2_{Q_{a\tau_m}} \\ &= \frac{1}{2}\|u_a(t)\|^2_{R_a} + \frac{1}{2}\|X(t)\|^2_{\widetilde{Q}_a}. \end{aligned} \tag{11}$$

Here, the matrices Q_{ijT} and Q_{ij}, $i, j \in \mathscr{P}$, $i \neq j$ are symmetric 2×2 matrices. Further, $\widetilde{Q}_{iT}$ and $\widetilde{Q}_i$ are $2N \times 2N$ symmetric matrices, with $N = n + 2$, and R_i are 2×2 symmetric and positive definite matrices for all $i \in \mathscr{P}$. The matrices $\widetilde{Q}_i$ and $\widetilde{Q}_{iT}$ have the similar structures except an additional subscript T, and are described as follows. $\widetilde{Q}_{\tau_j} = \left[\begin{array}{c|c} Q_1 & Q_2' \\ \hline Q_2 & Q_3 \end{array}\right]$, where $Q_1 = \text{diag}\{q_{\tau_1}, \ldots, q_{\tau_n}\}$ with $q_{\tau_i} = (\alpha_R Q_{\tau_j d} - Q_{\tau_j a})$ for $i = j$ and $q_{\tau_i} = 0$ for $i \neq j$, $Q_2 = [\hat{q}_1, \ldots, \hat{q}_n]$ with $\hat{q}_j = \begin{bmatrix} -\alpha_R Q_{\tau_j d} \\ Q_{\tau_j a} \end{bmatrix}$, $\hat{q}_l = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{0}_2 \end{bmatrix} \forall\, l \neq j$, and $Q_3 = \text{diag}\{\alpha_R Q_{\tau_j d}, -Q_{\tau_j a}\}$. $\widetilde{Q}_d = \left[\begin{array}{c|c} Q_4 & Q_5' \\ \hline Q_5 & Q_6 \end{array}\right]$, where $Q_4 = \alpha_R\ \text{diag}\{Q_{d\tau_1}, \ldots, Q_{d\tau_n}\}$, $Q_5 = -\alpha_R\begin{bmatrix} Q_{d\tau_1} & Q_{d\tau_2} & \cdots & Q_{d\tau_n} \\ \mathbf{0}_2 & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \end{bmatrix}$, and $Q_6 = \begin{bmatrix} (\alpha_R \sum_{j=1}^n Q_{d\tau_j} + \alpha_I Q_{da}) & -\alpha_I Q_{da} \\ -\alpha_I Q_{da} & \alpha_I Q_{da} \end{bmatrix}$. $\widetilde{Q}_a = \left[\begin{array}{c|c} Q_7 & Q_8' \\ \hline Q_8 & Q_9 \end{array}\right]$, where $Q_7 = \text{diag}\{q_{\tau_1}, \ldots, q_{\tau_n}\}$ with $q_{\tau_i} = Q_{a\tau_m}$ for $\tau_i = \tau_m$ and $q_{\tau_i} = \mathbf{0}_2$ for $\tau_i \neq \tau_m$; $Q_8 = [\hat{q}_{\tau_1}, \ldots, \hat{q}_{\tau_n}]$ with $\hat{q}_{\tau_i} = \begin{bmatrix} \mathbf{0}_2 \\ -Q_{a\tau_m} \end{bmatrix}$ for $\tau_i = \tau_m$, $\hat{q}_{\tau_i} = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{0}_2 \end{bmatrix}$ for $\tau_i \neq \tau_m$; and $Q_9 = \text{diag}\{\mathbf{0}_2, Q_{a\tau_m}\}$. The dimension of the matrices Q_3, Q_6, Q_9 is 4×4; Q_1, Q_4, Q_7 is $2n \times 2n$ and of Q_2, Q_5, Q_8 is $4 \times 2n$. The parameters $\alpha_R, \alpha_I \in \{0, 1\}$, reflect the fact that when the game is in rescue mode the parameters are set to $(\alpha_R, \alpha_I) = (1, 0)$, and they are set to $(\alpha_R, \alpha_I) = (0, 1)$ in the interception mode.

Remark 3: From (6) to (11), the parameters (α_R, α_I) appear only in the objectives of the defender and the targets and not the attacker. This implies that in our setting the attacker is oblivious to mode switching of the defender.

We assume that all the players are aware of the objectives of themselves as well as the other players and have access to the state vector $x(t)$ for all $t \in [t_0, t_0 + T]$. Each player $i \in \mathscr{P}$ solves the following optimal control problem:

$$\min_{u_i(.)} J_i\big(u_{\tau_1}(.), \ldots, u_{\tau_n}(.), u_d(.), u_a(.), t_0, T\big) \quad \text{s.t (3)}. \tag{12}$$

Due to linearity of the dynamics (3) and the quadratic nature of the performance metrics (5), problem (12) describes a $n + 2$ player finite horizon nonzero sum LQDG in rescue or interception mode. We seek to obtain the Nash equilibrium strategies of the players.

Definition 1 (Nash Equilibrium): The strategy profile $(u^*_{\tau_1}(.), u^*_{\tau_2}(.), \dots, u^*_{\tau_n}(.), u^*_d(.), u^*_a(.))$ is a Nash equilibrium for the LQDG (12) if the following set of inequalities hold true for every $i \in \mathscr{P}$:

$$J_i\big(u^*_i(.), u^*_{-i}(.), t_0, T\big) \le J_i\big(u_i(.), u^*_{-i}(.), t_0, T\big) \quad \forall u_i(.). \quad (13)$$

Here, the notation $-i$ stands for players other than i, that is $-i := \mathscr{P}\backslash\{i\}$, and $u_{-i}(.)$ is the strategy profile of all the players in $\mathscr{P}$ excluding player i.

Remark 4: It is possible to consider interactions involving cooperation between targets and the defender as a team playing against the attacker which will result in minmax strategies; see [2]. In our article, the defender's objectives are different in rescue and interception modes. Further, we do not assume cooperation between the targets and all the players individually minimize their objectives. So, we model the interactions as a nonzero sum game described by (12), and consider Nash equilibrium as the solution concept.

Remark 5: It is well known in differential games literature [33] that the strategies of players depend upon the information available to the players while taking their decisions; referred to as information structure. Commonly, two types of information structures are used in differential games. In the open-loop information structure, the decisions of players are functions of time and the initial condition. In the feedback information structure, the decisions of players are functions of the state variable. There exist methods for computing both the open-loop and feedback Nash equilibria [34]. However, in this article due to the complexity of analysis, to allow for defender's role switch, and later for using a receding horizon approach, we restrict our attention to open-loop information structure.

The open-loop Nash equilibrium strategies can be computed by jointly solving $N := n+2$ optimal control problems, given by (12) using the Pontryagin maximum principle. We thus have the following result from [34].

Theorem 1: [34, Th. 7.2] Consider the N player finite horizon LQDG described by (12) with open-loop information structure. Let there exist a solution set $\{P_i(t),\ i \in \mathscr{P}\}$ to the following N coupled Riccati differential equations:

$$\dot{P}_i(t) = P_i(t)\left(\sum_{j=1}^{n}\big[S_{\tau_j} P_{\tau_j}(t)\big] + S_d P_d(t) + S_a P_a(t)\right) - \widetilde{Q}_i \quad (14)$$

where $P_i(T) = \widetilde{Q}_{iT}$ and $S_i = B_i R_i^{-1} B_i'$. The unique open-loop Nash equilibrium solution at time $t \in [0, T]$ for every initial state $X(0)$ is given by

$$\begin{aligned} u^*_i(t; X(0)) &= -R_i^{-1} B_i' P_i(t)\Phi(t,0)X(0) \\ \dot{\Phi}(t,0) &= \left(-\sum_i S_i P_i(t)\right)\Phi(t,0) \\ &= A_{\text{cl}}(t)\Phi(t,0), \quad \Phi(0,0) = \mathbf{I}_{2N}. \end{aligned} \quad (15)$$

Upon using the open-loop Nash equilibrium strategies, the closed loop system matrix is given by $A_{\text{cl}}(t) = \big(-\sum_i S_i P_i(t)\big)$, and the closed-loop dynamic interaction environment (3) evolves according to

$$\dot{X}(t) = A_{\text{cl}}(t)X(t), \quad t \ge 0. \quad (16)$$

Remark 6: As we assumed that the matrix R_i is symmetric and positive definite, it can be easily verified that the cost function J_i given by (5) is strictly convex in $u_i(.)$ for all control functions $u_j(.)$ $j \ne i$ and for all the initial conditions X_0. This implies that the conditions obtained using the Pontryagin maximum principle are both necessary and sufficient.

Next, we discuss the conditions which guarantee the solvability of the coupled Riccati differential equation (14). Let us define

$$\begin{aligned} M &= \begin{bmatrix} \mathbf{0}_{2N} & -\mathbf{S} \\ -\mathbf{Q}' & \mathbf{0}_{2N^2} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_{\tau_1} & S_{\tau_2} & \cdots & S_{\tau_n} & S_d & S_a \end{bmatrix} \\ \mathbf{Q} &= \begin{bmatrix} \widetilde{Q}'_{\tau_1} & \widetilde{Q}'_{\tau_2} & \cdots & \widetilde{Q}'_{\tau_n} & \widetilde{Q}'_d & \widetilde{Q}'_a \end{bmatrix} \\ H(T) &= \begin{bmatrix} \mathbf{I}_{2N} & \mathbf{0}_{2N} & \cdots & \mathbf{0}_{2N} \end{bmatrix} e^{-MT} \\ &\quad \begin{bmatrix} \mathbf{I}_{2N} & \widetilde{Q}'_{\tau_1 T} & \cdots & \widetilde{Q}'_{\tau_n T} & \widetilde{Q}'_{dT} & \widetilde{Q}'_{aT} \end{bmatrix}'. \end{aligned} \quad (17)$$

The next result relates the solvability of the Riccati differential equations (14) with invertibility of the matrix $H(T)$. Here, $H(T), S_i, \widetilde{Q}_i$ and $\widetilde{Q}_{iT}$, $i \in \mathscr{P}$ are $2N \times 2N$ matrices, and M is a $2N(N+1) \times 2N(N+1)$ matrix.

Theorem 2: [34, Th. 7.1] For the N player finite horizon LQDG described by (12), the coupled Riccati differential equation (14) has a solution for every initial state $X(0)$ over the interval $[0, T]$ if and only if the matrix $H(T)$ is invertible.

The conditions under which the Riccati differential equation (14) admits a solution follow from [34, Proposition 7.6]. The state equation (16) can be solved as

$$\begin{aligned} X(t) &= \begin{bmatrix} \mathbf{I}_{2N} & \mathbf{0}_{2N} & \cdots & \mathbf{0}_{2N} \end{bmatrix} e^{M(t-T)} \\ &\quad \begin{bmatrix} \mathbf{I}_{2N} & \widetilde{Q}'_{\tau_{1T}} & \cdots & \widetilde{Q}'_{\tau_{nT}} & \widetilde{Q}'_{d_T} & \widetilde{Q}'_{a_T} \end{bmatrix}' H^{-1}(T)X_0. \end{aligned} \quad (18)$$

Remark 7: From Theorem (2) it is evident that the horizon length T must be selected such that the matrix $H(T)$ given by (17) is invertible; see also Remark 11.

IV. Switching Analysis Using Receding Horizon Approach

In Section III, we have analyzed the situation where the interactions of the players are fixed for a duration $[0, T]$ in one of the modes. Now, to allow for switching between the modes and for players to adapt their strategies, the open-loop Nash equilibrium solution is augmented with the *receding horizon* or moving horizon approach. In this method, every player computes the open-loop Nash equilibrium at each instant of time and implements the computed strategy for only one-time step. Players then repeat the procedure until the termination criteria are met while updating any change in the mode of the game (by the defender) and the closest target (by the attacker).

We now present the receding horizon approach for the N player game. We consider the policy or strategy time instants $t_k = k\delta$, $k = 0, 1, 2, \dots,$ with $t_0 = 0$ and $0 < \delta \ll T$. At any time instant t_k, using $X(t_k)$ as the initial state, players evaluate the open-loop Nash equilibrium control strategy over the planning horizon $[t_k, t_k + T]$, that is, players $i \in \mathscr{P}$ minimize the performance indices given by

$$J_i^{\mathrm{RH}} \triangleq J_i\big(u_{\tau_1}(.), \ldots, u_{\tau_n}(.), u_d(.), u_a(.), t_k, T\big). \quad (19)$$

The open-loop Nash equilibrium strategy of player i over the interval $[t_k, t_k + T]$ is obtained from (15). However, the open-loop Nash equilibrium strategies are implemented only for the period $[t_k, t_{k+1})$, and the receding horizon Nash control for player i at time $t \in [t_k, t_{k+1})$ with the initial state variable $X(t_k)$ is then given by

$$u_i^{\mathrm{RH}}(t; X(t_k)) = -R_i^{-1} B_i' P_i(t - t_k)\Phi(t - t_k, 0)X(t_k). \quad (20)$$

The state variable at time instant t_{k+1} is obtained from (18) as

$$X(t_{k+1}) = \begin{bmatrix} \mathbf{I}_{2N} & \mathbf{0}_{2N} & \cdots & \mathbf{0}_{2N} \end{bmatrix} e^{M(t_{k+1}-T)} \begin{bmatrix} \mathbf{I}_{2N} & \widetilde{Q}'_{\tau_1 T} & \cdots & \widetilde{Q}'_{\tau_n T} & \widetilde{Q}'_{d_T} & \widetilde{Q}'_{a_T} \end{bmatrix}' H^{-1}(T)X(t_k). \quad (21)$$

Next, at the time instant t_{k+1} the LQDG described by the objectives (19) and the dynamics (3) is solved by setting $t_k \to t_{k+1}$ and $X(t_k) \to X(t_{k+1})$ for the duration $[t_{k+1}, t_{k+1} + T]$. Again, the open-loop Nash equilibrium strategies of players, obtained similarly as (20), are implemented only for the period $[t_{k+1}, t_{k+2})$ to obtain the state variable $X(t_{k+2})$. This procedure is repeated again till the game termination criteria are met.

A. Target Update by the Attacker

The closest target pursued by the attacker can change with time requiring the attacker to update the closest target as the game proceeds in time. To incorporate this feature (F1), we assume that τ_m is the closest target to the attacker at time instant t_k, that is

$$\tau_m := \arg\min_{\tau \in \mathscr{T}} ||X_\tau(t_k) - X_a(t_k)||_2 \quad (22)$$

then the attacker pursues the target τ_m and plays the game described by the objectives (19) to obtain the open-loop Nash equilibrium strategies for the duration $[t_k, t_k+T]$. The attacker implements these strategies only for the duration $[t_k, t_{k+1})$. Then at the time instant t_{k+1} the attacker reevaluates the closest target to pursue, using (22), and computes the open-loop Nash equilibrium strategy for the duration $[t_{k+1}, t_{k+1} + T]$, and implements it for the duration $[t_{k+1}, t_{k+2})$. This process is repeated until the termination criterion is met.

Remark 8: It is possible that at the time instant t_k two or more targets could be at the closest distance to the attacker. To handle such a scenario, we assume that the attacker moves toward the target that is farthest from the defender. If these targets are equidistant from the defender then we assume that the attacker chooses one target randomly. However, when the attacker is equipped with a discriminatory sensor then the attacker chooses a target based on some criterion, for instance, minimum index.

B. Operational Mode Switch by the Defender

The defender can switch autonomously from rescue mode to interception mode and vice-versa, depending upon the state of the system; see feature (F2). We assume that the defender uses a switching function $\Psi : \mathbb{R}^n \to \mathbb{R}$ based on which the mode switching is realized at t_k, that is

$$(\alpha_R, \alpha_I) = \begin{cases} (0, 1), & \Psi(X(t_k)) \leq 0 \\ (1, 0), & \Psi(X(t_k)) > 0. \end{cases} \quad (23)$$

This implies, at time t_k, if $\Psi(X(t_k)) \leq 0$ then the parameters (α_R, α_I) in the objective functions (19) are set to $(0, 1)$ (interception mode), and if $\Psi(X(t_k)) > 0$ they are set to $(1, 0)$ (rescue mode). Once the operational mode is decided by the defender at time instant t_k, the open-loop Nash equilibrium strategies (20) are evaluated for the duration $[t_k, t_k + T]$ and implemented for the duration $[t_k, t_{k+1})$. Then, at the next time instant t_{k+1} the same procedure is repeated until termination criterion is met. As the state information is available to all the players, a distance-based criterion is a natural choice for the switching function. In this article, we consider the following switching function:

$$\Psi(X(t)) := ||X_a(t) - X_{\tau_m}(t)||_2 - \kappa\sigma_a. \quad (24)$$

This implies, when the distance between the attacker and the minimum distance target to the attacker, evaluated at time instant t_k, is less than or equal to $\kappa\sigma_a > 0$ with $\kappa \geq 1$, then the defender sets the operational mode as interception mode for the duration $[t_k, t_{k+1})$. The defender can implement these operational modes in coordination with the targets whenever a change in the sign of switching function (24) is observed, and this addresses feature (F2); see [35] probabilistic switching rules in the context of multiple vehicle intercept problem.

Remark 9: The parameter κ indicates the level of alertness of the defender. In other words, a highly alert defender reacts early to an attacker, who is approaching the target, by switching from rescue mode to interception mode.

Remark 10: We emphasize that in the receding horizon approach the interactions between the players remain fixed between the time instants t_k and t_{k+1}. This implies that players only require state information at the time instants t_k, $k = 0, 1, 2, \ldots,$ to implement the mode-dependent switching strategies.

Remark 11: The horizon length T indicates intertemporal decision-making behavior of the players. In (19), there is a tacit assumption that all the players use the same horizon length T while synthesizing their receding horizon strategies. Relaxing this assumption can increase the complexity in the analysis of switching strategies.

The receding horizon approach for obtaining switching strategies is presented in Algorithm 1. Here, in step 7 the function `minindex(.)` provides the target with minimum index; see also Remark 8.

V. Analysis of the Switching Strategies

In this section, we analyze the switching strategies obtained through the receding horizon approach and derive results related to the trajectories of the players.

Assumption 1: The matrices $Q_{ij} = Q_{ijT} = \mathbf{I}_2$ for all $i, j \in \mathscr{P}$, $i \neq j$ and $R_i = r_i\mathbf{I}_2$, $r_i > 0$, for $i \in \mathscr{P}$.

The above assumption implies that players minimize or maximize their Euclidean distances with other players, and

Algorithm 1 Synthesis of Switching Strategies Using Receding Horizon Approach

Data: R_i, $\widetilde{Q}_i$ and $\widetilde{Q}_{iT}$, $i \in \mathscr{P}$, κ, σ_d, σ_a, δ, T such that $H(T)$ is invertible.
Input: Initial locations $X_i(0),\ i \in \mathscr{P}$
Output: Outcome of the game: Rescue of all targets (or) capture of a target by the attacker (or) interception of the attacker by the defender.

Initialize $k = 0$, $t_0 = 0$, $X(t_0) = X(0)$ and the termination flag `tflag = 0`
`/* Iterate till the termination of the game */`
while *tflag=0* **do**
  $\mathscr{T}_1 := \arg\min_{\tau \in \mathscr{T}} ||X_\tau(t_k) - X_a(t_k)||_2$ `// Attacker choosing the minimum distance target`
  if $|\mathscr{T}_1| > 1$ **then**
    $\mathscr{S} := \arg\max_{a \in \mathscr{T}_1} ||X_\tau(t_k) - X_d(t_k)||_2$
    if $|\mathscr{S}| > 1$ **then**
      $\tau_m = \texttt{minindex}(\mathscr{S})$
    else
      $\tau_m = \mathscr{S}$
    end
  else
    $\tau_m = \mathscr{T}_1$
  end
  if $\Psi(X(t_k)) \le 0$ `// Game mode using (23)` **then**
    $(\alpha_R, \alpha_I) = (0, 1)$ `// Interception mode`
  else
    $(\alpha_R, \alpha_I) = (1, 0)$ `// Rescue mode`
  end
  Update $\widetilde{Q}_i$, $\widetilde{Q}_{iT}$ and compute $H(T)$ using (17)
  Solve the Ricatti differential equation (14) to obtain $P_i(t_k)$ for all $i \in \mathscr{P}$
  Set $t_{k+1} = t_k + \delta$
  Implement the open-loop Nash equilibrium strategies $u_i^{\mathrm{RH}}(t, X(t_k)),\ t \in [t_k, t_{k+1})$ using (20)
  Compute $X(t_{k+1})$ using (18)
  if $(\alpha_R, \alpha_I) = (1, 0)$ **then**
    if $||X_\tau(t_k) - X_d(t_k)||_2 \le \sigma_d\ \forall\ \tau \in \mathscr{T}$ *for* $t < T$ **then**
      Rescue of all targets by the defender
      Set `tflag=1`
    end
  end
  if $(\alpha_R, \alpha_I) = (0, 1)$ **then**
    if $||X_d(t_k) - X_a(t_k)||_2 \le \sigma_d$ *for* $t < T$ **then**
      Interception of the attacker by the defender
      Set `tflag=1`
    end
  end
  if $||X_{\tau_m}(t_k) - X_a(t_k)||_2 \le \sigma_a$ **then**
    Capture of the target by the attacker
    Set `tflag=1`
  end
  $k \leftarrow k + 1$
end

the penalties on the control efforts in x and y orientations are treated equally. In the interception mode, the attacker a is in direct conflict with its closest target τ_m, and is not responding to other targets in $\mathscr{T} \backslash \tau_m$ and the defender. Whereas the other targets in $\mathscr{T} \backslash \tau_m$ maximize their distances with the attacker and the defender d minimizes its distance with the attacker. This implies, it is sufficient to consider the interaction between the four players $\{\tau_m, \tau, d, a\}$ instead of $n + 2$ players $\{\mathscr{T}, d, a\}$. Based on this observation we have the following result.

Theorem 3: Let Assumption 1 hold true. Let t_k be the time instant when the game switches to the interception mode. Then, using the receding horizon strategies (20), the attacker a and its closest target τ_m move on the straight line joining their locations, evaluated at t_k, for the duration $[t_k, t_{k+1})$.

Proof: We consider the interaction between the players $\{\tau_m, \tau, d, a\}$. The Riccati differential equation (14) associated with player $i \in \{\tau_m, \tau, d, a\}$ is given by

$$\dot{P}_i = -\widetilde{Q}_i + P_i\left(S_{\tau_m} P_{\tau_m} + S_\tau P_\tau + S_d P_d + S_a P_a\right) \qquad (25)$$

where $P_i(t_k + T) = \widetilde{Q}_{iT}$. In the interception mode, the matrices entering the objective functions are

$$\widetilde{Q}_{\tau_m} = \widetilde{Q}_{\tau_m T} = \begin{bmatrix} -\mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 \end{bmatrix}$$

$$\widetilde{Q}_{\tau} = \widetilde{Q}_{\tau T} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & -\mathbf{I}_2 \end{bmatrix}$$

$$\widetilde{Q}_{d} = \widetilde{Q}_{dT} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & -\mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix}$$

and $\widetilde{Q}_a = \widetilde{Q}_{aT} = -\widetilde{Q}_{\tau_m}$. Then, it follows immediately that

$$\dot{P}_{\tau_m} + \dot{P}_a = \left(P_{\tau_m} + P_c\right)\left(S_{\tau_m} P_{\tau_m} + S_\tau P_\tau + S_d P_d + S_a P_a\right)$$

with $P_{\tau_m}(t_k + T) + P_a(t_k + T) = \mathbf{0}_8$. This implies that

$$P_{\tau_m}(t) + P_a(t) = \mathbf{0}_8 \Rightarrow P_{\tau_m}(t) = -P_a(t), \quad t \in [t_k, t_k + T]. \qquad (26)$$

We partition the matrix $P_i(t)$ for $i \in \{\tau_m, \tau, d, a\}$ as

$$P_i(t) = \begin{bmatrix} P_i^{11} & P_i^{12} & P_i^{13} & P_i^{14} \\ P_i^{21} & P_i^{22} & P_i^{23} & P_i^{24} \\ P_i^{31} & P_i^{32} & P_i^{33} & P_i^{34} \\ P_i^{41} & P_i^{42} & P_i^{43} & P_i^{44} \end{bmatrix}. \qquad (27)$$

We denote by $\Gamma_1(t) := S_{\tau_m} P_{\tau_m}(t) + S_\tau P_\tau(t) + S_d P_d(t) + S_a P_a(t)$. Substituting for $S_i = B_i R_i^{-1} B_i'$ and $R_i = r_i \mathbf{I}_2$ for $i \in \{\tau_m, \tau, d, a\}$ we obtain

$$\Gamma_1(t) = \begin{bmatrix} r_{\tau_m}^{-1} P_{\tau_m}^{11} & r_{\tau_m}^{-1} P_{\tau_m}^{12} & r_{\tau_m}^{-1} P_{\tau_m}^{13} & r_{\tau_m}^{-1} P_{\tau_m}^{14} \\ r_\tau^{-1} P_\tau^{21} & r_\tau^{-1} P_\tau^{22} & r_\tau^{-1} P_\tau^{23} & r_\tau^{-1} P_\tau^{24} \\ r_d^{-1} P_d^{31} & r_d^{-1} P_d^{32} & r_d^{-1} P_d^{33} & r_d^{-1} P_d^{34} \\ -r_c^{-1} P_{\tau_m}^{41} & -r_c^{-1} P_{\tau_m}^{42} & -r_c^{-1} P_{\tau_m}^{43} & -r_c^{-1} P_{\tau_m}^{44} \end{bmatrix}. \qquad (28)$$

Using (27) in (25) for $i = \tau_m$, and premultiplying with the matrix $\begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix}$ we obtain

$$\begin{bmatrix} \sum_{l=1}^{4} \dot{P}_{\tau_m}^{l1} & \sum_{l=1}^{4} \dot{P}_{\tau_m}^{l2} & \sum_{l=1}^{4} \dot{P}_{\tau_m}^{l3} & \sum_{l=1}^{4} \dot{P}_{\tau_m}^{l4} \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^{4} P_{\tau_m}^{l1} & \sum_{l=1}^{4} P_{\tau_m}^{l2} & \sum_{l=1}^{4} P_{\tau_m}^{l3} & \sum_{l=1}^{4} P_{\tau_m}^{l4} \end{bmatrix} \Gamma_1(t) \quad (29)$$

where $\sum_{l=1}^{4} P_{\tau_m}^{l1}(t_k+T) = \sum_{l=1}^{4} P_{\tau_m}^{l2}(t_k+T) = \sum_{l=1}^{4} P_{\tau_m}^{l3}(t_k+T) = \sum_{l=1}^{4} P_{\tau_m}^{l4}(t_k+T) = \mathbf{0}_2$. This implies that

$$P_{\tau_m}^{1j}(t) + P_{\tau_m}^{2j}(t) + P_{\tau_m}^{3j}(t) + P_{\tau_m}^{4j}(t) = \mathbf{0}_2 \quad (30)$$

for all $t \in [t_k, t_k + T]$ for every $j \in \{1, 2, 3, 4\}$. Again, premultiplying (25) with $\begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}$ and repeating the same analysis as before we obtain

$$P_{\tau_m}^{1j}(t) + P_{\tau_m}^{4j}(t) = \mathbf{0}_2 \Rightarrow P_{\tau_m}^{1j}(t) = -P_{\tau_m}^{4j}(t) \quad (31)$$

for all $t \in [t_k, t_k + T]$ for every $j \in \{1, 2, 3, 4\}$. Using (31) in (30) we have

$$P_{\tau_m}^{2j}(t) + P_{\tau_m}^{3j}(t) = \mathbf{0}_2 \Rightarrow P_{\tau_m}^{2j}(t) = -P_{\tau_m}^{3j}(t) \quad (32)$$

for all $t \in [t_k, t_k+T]$ for every $j \in \{1, 2, 3, 4\}$. Again, using (27) in (25) for $i = \tau_m$, and postmultiplying with the matrix $\begin{bmatrix} \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{bmatrix}'$ we obtain

$$\begin{bmatrix} \sum_{l=1}^{4} \dot{P}_{\tau_m}^{1l} \\ \sum_{l=1}^{4} \dot{P}_{\tau_m}^{2l} \\ \sum_{l=1}^{4} \dot{P}_{\tau_m}^{3l} \\ \sum_{l=1}^{4} \dot{P}_{\tau_m}^{4l} \end{bmatrix} = P_{\tau_m} \begin{bmatrix} r_{\tau_m}^{-1} \sum_{l=1}^{4} P_{\tau_m}^{1l} \\ r_{\tau}^{-1} \sum_{l=1}^{4} P_{\tau}^{2l} \\ r_d^{-1} \sum_{l=1}^{4} P_d^{3l} \\ -r_a^{-1} \sum_{l=1}^{4} P_{\tau_m}^{4l} \end{bmatrix}, \quad \begin{bmatrix} \sum_{l=1}^{4} P_{\tau_m}^{1l}(t_k+T) \\ \sum_{l=1}^{4} P_{\tau_m}^{2l}(t_k+T) \\ \sum_{l=1}^{4} P_{\tau_m}^{3l}(t_k+T) \\ \sum_{l=1}^{4} P_{\tau_m}^{4l}(t_k+T) \end{bmatrix} = \mathbf{0}_{8\times 2}. \quad (33)$$

Repeating the above exercise for the matrices P_τ and P_a and then rearranging terms we obtain

$$\begin{bmatrix} \sum_{l=1}^{4} \dot{P}_{\tau_m}^{1l} \\ \sum_{l=1}^{4} \dot{P}_{\tau}^{2l} \\ \sum_{l=1}^{4} \dot{P}_{d}^{3l} \\ \sum_{l=1}^{4} \dot{P}_{\tau_m}^{4l} \end{bmatrix} = \Xi(t) \begin{bmatrix} \sum_{l=1}^{4} P_{\tau_m}^{1l} \\ \sum_{l=1}^{4} P_{\tau}^{2l} \\ \sum_{l=1}^{4} P_{d}^{3l} \\ \sum_{l=1}^{4} P_{\tau_m}^{4l} \end{bmatrix}, \quad \begin{bmatrix} \sum_{l=1}^{4} P_{\tau_m}^{1l}(t_k+T) \\ \sum_{l=1}^{4} P_{\tau}^{2l}(t_k+T) \\ \sum_{l=1}^{4} P_{d}^{3l}(t_k+T) \\ \sum_{l=1}^{4} P_{\tau_m}^{4l}(t_k+T) \end{bmatrix} = \mathbf{0}_{8\times 2}$$

where

$$\Xi(t) = \begin{bmatrix} r_{\tau_m}^{-1} P_{\tau_m}^{11} & r_{\tau}^{-1} P_{\tau}^{12} & r_d^{-1} P_d^{13} & -r_a^{-1} P_{\tau_m}^{14} \\ r_{\tau_m}^{-1} P_{\tau_m}^{21} & r_{\tau}^{-1} P_{\tau}^{22} & r_d^{-1} P_d^{23} & -r_a^{-1} P_{\tau_m}^{24} \\ r_{\tau_m}^{-1} P_{\tau_m}^{31} & r_{\tau}^{-1} P_{\tau}^{32} & r_d^{-1} P_d^{33} & -r_a^{-1} P_{\tau_m}^{34} \\ r_{\tau_m}^{-1} P_{\tau_m}^{41} & r_{\tau}^{-1} P_{\tau}^{42} & r_d^{-1} P_d^{43} & -r_a^{-1} P_{\tau_m}^{44} \end{bmatrix}. \quad (34)$$

This implies that

$$\sum_{l=1}^{4} P_{\tau_m}^{1l}(t) = \sum_{l=1}^{4} P_{\tau}^{2l}(t) = \sum_{l=1}^{4} P_{d}^{3l}(t) = \sum_{l=1}^{4} P_{\tau_m}^{4l}(t) = \mathbf{0}_2 \quad (35)$$

for all $t \in [t_k, t_k + T]$. Next, using (35) in (33) we also have that

$$\sum_{l=1}^{4} P_{\tau_m}^{2l}(t) = \sum_{l=1}^{4} P_{\tau_m}^{3l}(t) = \mathbf{0}_2 \quad (36)$$

for all $t \in [t_k, t_k+T]$. Next, using (31) we analyze the elements $P_{\tau_m}^{12}$ and $P_{\tau_m}^{13}$

$$\begin{aligned} \dot{P}_{\tau_m}^{12} &= r_{\tau_m}^{-1} P_{\tau_m}^{11} P_{\tau_m}^{12} + r_{\tau}^{-1} P_{\tau_m}^{12} P_{\tau}^{22} + r_d^{-1} P_{\tau_m}^{13} P_d^{32} - r_a^{-1} P_{\tau_m}^{14} P_{\tau_m}^{42} \\ &= \left(r_{\tau_m}^{-1} P_{\tau_m}^{11} + r_a^{-1} P_{\tau_m}^{14}\right) P_{\tau_m}^{12} + r_{\tau}^{-1} P_{\tau_m}^{12} P_{\tau}^{22} + r_d^{-1} P_{\tau_m}^{13} P_d^{32} \\ \dot{P}_{\tau_m}^{13} &= r_{\tau_m}^{-1} P_{\tau_m}^{11} P_{\tau_m}^{13} + r_{\tau}^{-1} P_{\tau_m}^{12} P_{\tau}^{23} + r_d^{-1} P_{\tau_m}^{13} P_d^{33} - r_a^{-1} P_{\tau_m}^{14} P_{\tau_m}^{43} \\ &= \left(r_{\tau_m}^{-1} P_{\tau_m}^{11} + r_a^{-1} P_{\tau_m}^{14}\right) P_{\tau_m}^{13} + r_{\tau}^{-1} P_{\tau_m}^{12} P_{\tau}^{23} + r_d^{-1} P_{\tau_m}^{13} P_d^{33} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \dot{P}_{\tau_m}^{12} & \dot{P}_{\tau_m}^{13} \end{bmatrix} &= \left(r_{\tau_m}^{-1} P_{\tau_m}^{11} + r_a^{-1} P_{\tau_m}^{14}\right) \begin{bmatrix} P_{\tau_m}^{12} & P_{\tau_m}^{13} \end{bmatrix} \\ &\quad + \begin{bmatrix} P_{\tau_m}^{12} & P_{\tau_m}^{13} \end{bmatrix} \begin{bmatrix} r_{\tau}^{-1} P_{\tau}^{22} & r_{\tau}^{-1} P_{\tau}^{23} \\ r_d^{-1} P_d^{32} & r_d^{-1} P_d^{33} \end{bmatrix}. \end{aligned}$$

The terminal conditions are $P_{\tau_m}^{12}(t_k + T) = \mathbf{0}_2$ and $P_{\tau_m}^{13}(t_k + T) = \mathbf{0}_2$. From the matrix variation of constants formula [36, Th. 1, p. 59] it follows immediately that $P_{\tau_m}^{12}(t) = P_{\tau_m}^{13}(t) = \mathbf{0}_2$ for all $t \in [t_k, t_k + T]$. Then from (31) we have

$$P_{\tau_m}^{12}(t) = P_{\tau_m}^{13}(t) = P_{\tau_m}^{42}(t) = P_{\tau_m}^{43}(t) = \mathbf{0}_2, \quad t \in [t_k, t_k + T]. \quad (37)$$

Next, using (37) in (25) with $i = \tau_m$, we have

$$\begin{bmatrix} \dot{P}_{\tau_m}^{22} & \dot{P}_{\tau_m}^{23} \end{bmatrix} = \begin{bmatrix} P_{\tau_m}^{22} & P_{\tau_m}^{23} \end{bmatrix} \begin{bmatrix} r_{\tau}^{-1} P_{\tau}^{22} & r_{\tau}^{-1} P_{\tau}^{23} \\ r_d^{-1} P_d^{32} & r_d^{-1} P_d^{33} \end{bmatrix} \quad (38)$$

with $P_{\tau_m}^{22}(t_k + T) = P_{\tau_m}^{23}(t_k + T) = \mathbf{0}_2$. This coupled with (32) implies

$$P_{\tau_m}^{22}(t) = P_{\tau_m}^{23}(t) = P_{\tau_m}^{32}(t) = P_{\tau_m}^{33}(t) = \mathbf{0}_2, \quad t \in [t_k, t_k + T]. \quad (39)$$

Again, using (39) in (25) with $i = \tau_m$ we have

$$\begin{bmatrix} \dot{P}_{\tau_m}^{21} & \dot{P}_{\tau_m}^{24} \end{bmatrix} = \begin{bmatrix} P_{\tau_m}^{21} & P_{\tau_m}^{24} \end{bmatrix} \begin{bmatrix} r_{\tau_m}^{-1} P_{\tau_m}^{11} & r_{\tau_m}^{-1} P_{\tau_m}^{14} \\ -r_a^{-1} P_{\tau_m}^{41} & -r_a^{-1} P_{\tau_m}^{44} \end{bmatrix} \quad (40)$$

with $P_{\tau_m}^{21}(t_k + T) = P_{\tau_m}^{24}(t_k + T) = \mathbf{0}_2$. This coupled with (32) implies

$$P_{\tau_m}^{21}(t) = P_{\tau_m}^{24}(t) = P_{\tau_m}^{31}(t) = P_{\tau_m}^{34}(t) = \mathbf{0}_2, \quad t \in [t_k, t_k + T]. \quad (41)$$

Thus the structure of matrix P_{τ_m} is given by

$$P_{\tau_m}(t) = -P_a(t) = \begin{bmatrix} -K(t) & \mathbf{0}_2 & \mathbf{0}_2 & K(t) \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ K(t) & \mathbf{0}_2 & \mathbf{0}_2 & -K(t) \end{bmatrix}. \quad (42)$$

where $K(t) = \begin{bmatrix} k_1(t) & k_2(t) \\ k_3(t) & k_4(t) \end{bmatrix}_{2\times 2}$, $k_i(t) \in \mathbb{R}$, $i = 1, 2, 3, 4$. Next, writing $P^{11}_{\tau_m} = -K$ in (25) we get

$$\dot{K} = -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)KK - \mathbf{I}_2, \quad K(t_k + T) = \mathbf{I}_2. \tag{43}$$

Now, expanding K we have

$$\begin{aligned}\dot{k}_1 &= -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)\left(k_1^2 + k_2k_3\right) - 1, \quad k_1(t_k + T) = 1\\ \dot{k}_2 &= -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)(k_1 + k_4)k_2, \quad k_2(t_k + T) = 0\\ \dot{k}_3 &= -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)(k_1 + k_4)k_3, \quad k_3(t_k + T) = 0\\ \dot{k}_4 &= -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)\left(k_3k_2 + k_4^2\right) - 1, \quad k_4(t_k + T) = 1.\end{aligned}$$

Notice, $k_2(t)$ and $k_3(t)$ are solutions of the differential equation

$$\dot{\gamma} = -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)(k_1 + k_4)\gamma, \quad \gamma(t_k + T) = 0.$$

This implies that $k_2(t) = k_3(t) = 0$ for all $t \in [t_k, t_k + T]$. Next, $k_1(t)$ and $k_4(t)$ satisfy the differential equation

$$\dot{\zeta}_1 = -\left(r^{-1}_{\tau_m} - r^{-1}_a\right)\zeta_1^2 - 1, \quad \zeta_1(t_k + T) = 1. \tag{44}$$

So, we have that $k_1(t) = k_4(t) = \zeta_1(t)$ for all $t \in [t_k, t_k+T]$. So, we have

$$P_{\tau_m}(t) = -P_a(t) = \zeta_1(t)\begin{bmatrix} -\mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2\\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2\\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2\\ \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2\end{bmatrix}. \tag{45}$$

Now, using the open-loop Nash equilibrium strategies (15) in the game with players $\{\tau_m, \tau, d, a\}$ the state variables of the target τ_m and the attacker a are obtained, from (16), as

$$\begin{bmatrix}\dot{X}_{\tau_m}\\ \dot{X}_a\end{bmatrix} = -\zeta_1(t)\begin{bmatrix}R^{-1}_{\tau_m}\\ R^{-1}_a\end{bmatrix}\left(X_a - X_{\tau_m}\right). \tag{46}$$

Using the above we have

$$\begin{aligned}\dot{X}_a - \dot{X}_{\tau_m} &= -\zeta_1(t)\left(R^{-1}_a - R^{-1}_{\tau_m}\right)\left(X_a - X_{\tau_m}\right)\\ &= \left(\frac{r_a - r_{\tau_m}}{r_a r_{\tau_m}}\right)\zeta_1(t)\left(X_a - X_{\tau_m}\right).\end{aligned} \tag{47}$$

Representing the x and y coordinates of $X_a - X_{\tau_m}$ as $z_1 := x_a - x_{\tau_m}$ and $z_2 := y_a - y_{\tau_m}$, the above equation can be written as

$$\dot{z}_1 = \left(\frac{r_a - r_{\tau_m}}{r_a r_{\tau_m}}\right)\zeta_1(t)z_1, \quad \dot{z}_2 = \left(\frac{r_a - r_{\tau_m}}{r_a r_{\tau_m}}\right)\zeta_1(t)z_2. \tag{48}$$

Slope of the line joining the attacker a and the target τ_m at time t is given by $s_1(t) = (y_a(t) - y_{\tau_m}(t))/(x_a(t) - x_{\tau_m}(t)) = (z_2(t))/(z_1(t))$, $z_1(t) \neq 0$. From (48) we have that when $z_1(t_k) \neq 0$ then $z_1(t) \neq 0$ for all $t \in [t_k, t_k + T]$. The time derivative of the slope $s_1(t)$ results in

$$\begin{aligned}\dot{s}_1(t) &= \frac{\dot{z}_2(t)z_1(t) - \dot{z}_1(t)z_2(t)}{z_1^2(t)}\\ &= \left(\frac{r_a - r_{\tau_m}}{r_{\tau_m} r_a}\right)\left(\frac{z_1(t)z_2(t) - z_1(t)z_2(t)}{z_1^2(t)}\right)\zeta_1(t) = 0.\end{aligned}$$

Clearly, this implies when $x_a(t_k) \neq x_{\tau_m}(t_k)$ the slope $s_1(t) = s_1(t_k)$ for all $t \in [t_k, t_{k+1})$. When $z_1(t_k) = x_a(t_k) - x_{\tau_m}(t_k) = 0$ then $x_a(t) = x_{\tau_m}(t)$ for all t, this implies the attacker a and the target τ_m continue along the y-axis for all $t \in [t_k, t_{k+1})$. ■

Next, we have the following result to infer about the geometric structure of trajectories of the targets $\tau \in \mathscr{T}\backslash\tau_m$.

Theorem 4: Let Assumption 1 hold true. Let t_k be the time instant when the game switches to the interception mode. Let $r_\tau = r_{\tau_i}$ for all $\tau, \tau_i \in \mathscr{T}$ and $\tau_i \neq \tau$. Then the line joining the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$ evolves with a constant slope for the time duration $[t_k, t_{k+1})$.

Proof: Using the open loop Nash controls, (42) and (28), the state vector is written as

$$\begin{bmatrix}\dot{X}_{\tau_m}\\ \dot{X}_\tau\\ \dot{X}_d\\ \dot{X}_a\end{bmatrix} = -\begin{bmatrix} -r^{-1}_{\tau_m}K(t) & \mathbf{0}_2 & \mathbf{0}_2 & r^{-1}_{\tau_m}K(t)\\ r^{-1}_\tau P^{21}_\tau & r^{-1}_\tau P^{22}_\tau & r^{-1}_\tau P^{23}_\tau & r^{-1}_\tau P^{24}_\tau\\ r^{-1}_d P^{31}_d & r^{-1}_d P^{32}_d & r^{-1}_d P^{33}_d & r^{-1}_d P^{34}_d\\ -r^{-1}_a K(t) & \mathbf{0}_2 & \mathbf{0}_2 & r^{-1}_a K(t)\end{bmatrix}\begin{bmatrix}X_{\tau_m}\\ X_\tau\\ X_d\\ X_a\end{bmatrix}.$$

Then the position vectors of τ_m, τ, and a satisfy

$$\begin{aligned}\dot{X}_{\tau_m} &= r^{-1}_{\tau_m}K(t)\left(X_{\tau_m} - X_a\right)\\ \dot{X}_\tau &= -r^{-1}_\tau\left[P^{21}_\tau X_{\tau_m} + P^{22}_\tau X_\tau + P^{23}_\tau X_d + P^{24}_\tau X_a\right]\\ \dot{X}_a &= r^{-1}_a K(t)\left(X_{\tau_m} - X_a\right).\end{aligned}$$

Using the above, we have

$$\begin{aligned}\dot{X}_{\tau_m} - \dot{X}_\tau &= r^{-1}_{\tau_m}K(t)\left(X_{\tau_m} - X_a\right)\\ &\quad + r^{-1}_\tau\left[P^{21}_\tau X_{\tau_m} + P^{22}_\tau X_\tau + P^{23}_\tau X_d + P^{24}_\tau X_a\right].\end{aligned} \tag{49}$$

Next, using (25) and (27) for $i = \tau$, we have

$$\begin{aligned}\dot{P}^{23}_\tau &= r^{-1}_{\tau_m}P^{21}_\tau P^{13}_{\tau_m} + r^{-1}_\tau P^{22}_\tau P^{23}_\tau\\ &\quad + r^{-1}_d P^{23}_\tau P^{33}_d - r^{-1}_a P^{24}_\tau P^{43}_{\tau_m}, \; P^{23}_\tau(t_k + T) = \mathbf{0}_2.\end{aligned} \tag{50}$$

Using (42), we can further reduce (50) to the following:

$$\dot{P}^{23}_\tau = \left(r^{-1}_\tau P^{22}_\tau\right)P^{23}_\tau + P^{23}_\tau\left(r^{-1}_d P^{33}_d\right), \quad P^{23}_\tau(t_k + T) = \mathbf{0}_2. \tag{51}$$

From the matrix variation of constants formula [36, Th. 1, p. 59] it follows immediately that:

$$P^{23}_\tau(t) = \mathbf{0}_2, \quad t \in [t_k, t_k + T]. \tag{52}$$

Similarly, (25) and applying (27) for $i = \tau, \tau_m$, we have

$$\begin{aligned}\dot{P}^{24}_\tau &= -\mathbf{I}_2 + r^{-1}_{\tau_m}P^{21}_\tau P^{14}_{\tau_m} + r^{-1}_\tau P^{22}_\tau P^{24}_\tau + r^{-1}_d P^{23}_\tau P^{34}_d\\ &\quad - r^{-1}_a P^{24}_\tau P^{44}_{\tau_m}, \quad P^{24}_\tau(t_k + T) = \mathbf{I}_2\end{aligned} \tag{53}$$

$$\begin{aligned}\dot{P}^{11}_{\tau_m} &= \mathbf{I}_2 + r^{-1}_{\tau_m}P^{11}_{\tau_m}P^{11}_{\tau_m} + r^{-1}_\tau P^{12}_{\tau_m}P^{21}_\tau + r^{-1}_d P^{13}_{\tau_m}P^{31}_d\\ &\quad - r^{-1}_a P^{14}_{\tau_m}P^{41}_{\tau_m}, \quad P^{11}_{\tau_m}(t_k + T) = -\mathbf{I}_2.\end{aligned} \tag{54}$$

Using $P^{11}_{\tau_m} = -K(t)$, $P^{23}_\tau = \mathbf{0}_2$, and $r_{\tau_m} = r_\tau$ we can further reduce (53) and (54) as

$$\begin{aligned}\dot{P}^{24}_\tau - \dot{K}(t) &= r^{-1}_\tau\left[P^{21}_\tau K(t) + P^{22}_\tau P^{24}_\tau + K(t)K(t)\right]\\ &\quad + r^{-1}_a\left[P^{24}_\tau K(t) - K(t)K(t)\right]\end{aligned} \tag{55}$$

with $P^{24}_\tau(t_k + T) - K(t_k + T) = \mathbf{0}_2$. From (35) and (52) we have

$$P^{21}_\tau + P^{22}_\tau + P^{23}_\tau + P^{24}_\tau = \mathbf{0}_2 \Rightarrow P^{21}_\tau = -P^{22}_\tau - P^{24}_\tau.$$

Using this in (55) we get

$$\begin{aligned}\dot{P}^{24}_\tau - \dot{K}(t) &= \left[r^{-1}_\tau P^{22}_\tau\right]\left(P^{24}_\tau - K(t)\right)\\ &\quad + \left(P^{24}_\tau - K(t)\right)\left[\left(r^{-1}_a - r^{-1}_\tau\right)K(t)\right]\end{aligned} \tag{56}$$

with $P^{24}_\tau(t_k + T) - K(t_k + T) = \mathbf{0}_2$.

Again, from the matrix variation of constants formula [36, Th. 1, p. 59] it follows immediately that

$$P_\tau^{24} = K(t), \quad t \in [t_k, t_k + T]. \tag{57}$$

Using (27) for $i = \tau$ in (25), we have that P_τ^{22} satisfies

$$\dot{P}_\tau^{22} = r_\tau^{-1} P_\tau^{22} P_\tau^{22} + \mathbf{I}_2, \quad P_\tau^{22}(t_k + T) = -\mathbf{I}_2. \tag{58}$$

We solve (58) using the same approach as in solving (43) to get $P_\tau^{22}(t) = \zeta_2(t)\mathbf{I}_2$ where $\zeta_2(t)$ satisfies the differential equation

$$\dot{\zeta}_2(t) = r_\tau^{-1}\zeta_2^2(t) + 1, \quad \zeta_2(t_k + T) = -1. \tag{59}$$

Using the above, (49) can be written as

$$\begin{aligned}
&\dot{X}_{\tau_m} - \dot{X}_\tau \\
&= r_\tau^{-1} K(t)\big(X_{\tau_m} - X_a\big) \\
&\quad + r_\tau^{-1}\big[\big(-P_\tau^{22} - P_\tau^{23} - P_\tau^{24}\big)X_{\tau_m} + P_\tau^{22} X_\tau + K(t)X_a\big] \\
&= -r_\tau^{-1} P_\tau^{22}\big(X_{\tau_m} - X_\tau\big) \\
&= -r_\tau^{-1}\zeta_2(t)\big(X_{\tau_m} - X_\tau\big).
\end{aligned} \tag{60}$$

Representing the x and y coordinates of $X_{\tau_m} - X_\tau$ as $z_1 := x_{\tau_m} - x_\tau$ and $z_2 := y_{\tau_m} - y_\tau$, then (60) can be written as

$$\dot{z}_1 = -r_\tau^{-1}\zeta_2 z_1, \quad \dot{z}_2 = -r_\tau^{-1}\zeta_2 z_2. \tag{61}$$

Slope of the line joining the target τ_m and the target τ at time t is given by $s_2(t) = (y_{\tau_m}(t) - y_\tau(t))/(x_{\tau_m}(t) - x_\tau(t)) = (z_2(t))/(z_1(t))$, $z_1(t) \neq 0$. From (61) we have that when $z_1(t_k) \neq 0$ then $z_1(t) \neq 0$ for all $t \in [t_k, t_k + T]$. The time derivative of the slope $s_2(t)$ results in

$$\begin{aligned}
\dot{s}_2(t) &= \frac{\dot{z}_2(t)z_1(t) - \dot{z}_1(t)z_2(t)}{z_1^2(t)} \\
&= -r_\tau^{-1}\left(\frac{z_1(t)z_2(t) - z_1(t)z_2(t)}{z_1^2(t)}\right)\zeta_2(t) = 0.
\end{aligned}$$

Clearly, this implies when $x_{\tau_m}(t_k) \neq x_\tau(t_k)$ the slope $s_2(t) = s_2(t_k)$ for all $t \in [t_k, t_{k+1})$. When $z_1(t_k) = x_{\tau_m}(t_k) - x_\tau(t_k) = 0$ then $x_{\tau_m}(t) = x_\tau(t)$ for all t, this implies the target τ_m and the target τ continue along the y-axis for all $t \in [t_k, t_{k+1})$. ■

Theorem 4 says that when all the targets are identical, that is, with $r_\tau = r_{\tau_i}$ for all $\tau, \tau_i \in \mathscr{T}$ and $\tau_i \neq \tau$, then the line joining the minimum distance target τ_m and other targets $\mathscr{T}\backslash\{\tau_m\}$ evolves with a constant slope. Using the two previous results we have the following corollary.

Corollary 1: Let Assumption 1 hold true. Let t_k be the time instant when the game switches to the interception mode. Let $r_{\tau_i} = r_{\tau_j}$ for all $\tau_i, \tau_j \in \mathscr{T}$ and $i \neq j$. Then the angle between the lines joining the attacker (a), the minimum distance target (τ_m) and a target $\tau \in \mathscr{T}\backslash\tau_m$ remains constant for the duration $[t_k, t_{k+1})$.

Proof: From Theorem 3 and Theorem 4 we know that slopes of the line joining the players a and τ_m, and the line joining τ_m and $\tau \in \mathscr{T}$ remain constant during the execution period $[t_k, t_{k+1})$. The statement of the theorem follows immediately from this observation. ■

From Theorem 3, the attacker a and the target τ_m move in a straight line till the next time instant t_{k+1}. Now, at t_{k+1} it is possible that the minimum distance target τ_m (at time t_k) is no longer at a minimum distance to a as other targets in $\mathscr{T}\backslash\tau_m$ are trying to maximize their distance with a. In the following we derive conditions under which the target τ_m continues to be at a minimum distance to the attacker for the entire time duration $[t_k, t_{k+1})$. Toward this end, we provide some auxiliary results. Let us denote by $d_1 := X_{\tau_m} - X_a$ and $d_2 := X_{\tau_m} - X_\tau$. We have the following assumption on the decision horizon and the penalty parameters.

Lemma 1: Let Assumption 1 hold true. Let t_k be the time instant when the game switches to the interception mode. Let us assume $d_1(t_k) \neq 0$ and $d_2(t_k) \neq 0$. Let $r_{\tau_i} = r_{\tau_j}$ for all $\tau_i, \tau_j \in \mathscr{T}$ and $i \neq j$, and the penalty parameters of a target $\tau \in \mathscr{T}$ and the attacker a satisfy the condition $0 < (r_\tau - r_a)/(r_\tau r_a) < 1$. Then, the distance between the attacker a and it's minimum distance target τ_m, decreases with time for the time duration $[t_k, t_{k+1})$. Further, with $t_{k+1} - t_k = \delta$,

1) if the length of the planning horizon $T > 0$ satisfies

$$\begin{aligned}
&\sqrt{r_\tau}\left[k\pi - \tan^{-1}\left(\frac{1}{\sqrt{r_\tau}}\right)\right] + \delta < T \\
&\quad < \sqrt{r_\tau}\left[\left(k + \frac{1}{2}\right)\pi - \tan^{-1}\left(\frac{1}{\sqrt{r_\tau}}\right)\right] + \delta, \quad k \in \mathbb{Z}
\end{aligned} \tag{62}$$

then distance between the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$, increases with time for the time duration $[t_k, t_{k+1})$,

2) if the length of the planning horizon $T > 0$ satisfies

$$\begin{aligned}
&\sqrt{r_\tau}\left[\left(k - \frac{1}{2}\right)\pi - \tan^{-1}\left(\frac{1}{\sqrt{r_\tau}}\right)\right] + \delta \\
&\quad < T < \sqrt{r_\tau}\left[k\pi - \tan^{-1}\left(\frac{1}{\sqrt{r_\tau}}\right)\right] + \delta, \quad k \in \mathbb{Z}
\end{aligned} \tag{63}$$

then distance between the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$, decreases with time for the time duration $[t_k, t_{k+1})$.

Proof: Equations (47) and (60) are given by

$$\dot{d}_1 = -\frac{r_\tau - r_a}{r_\tau r_a}\zeta_1 d_1, \quad \dot{d}_2 = -\frac{1}{r_\tau}\zeta_2 d_2. \tag{64}$$

Next, consider the functions $V_1(t) = (1/2)d_1'(t)d_1(t)$ and $V_2(t) = (1/2)d_2'(t)\, d_2(t)$ defined over the time duration $[t_k, t_{k+1})$. Clearly, $V_1(t) \geq 0$ and $V_2(t) \geq 0$ for all $t \in [t_k, t_k + T]$. The differential (44) and (59) are solved [37] as

$$\zeta_1(t) = \begin{cases}
\sqrt{\dfrac{r_a r_\tau}{r_a - r_\tau}}\tan\left(\dfrac{t_k + T - t}{\sqrt{\frac{r_a r_\tau}{r_a - r_\tau}}} + \tan^{-1}\left(\sqrt{\dfrac{r_a - r_\tau}{r_a r_\tau}}\right)\right) & r_a > r_\tau \\
\sqrt{\dfrac{r_a r_\tau}{r_\tau - r_a}}\tanh\left(\dfrac{t_k + T - t}{\sqrt{\frac{r_a r_\tau}{r_\tau - r_a}}} + \tanh^{-1}\left(\sqrt{\dfrac{r_\tau - r_a}{r_a r_\tau}}\right)\right) & r_\tau > r_a
\end{cases}$$

$$\zeta_2(t) = -\sqrt{r_\tau}\tan\left(\frac{t_k + T - t}{\sqrt{r_\tau}} + \tan^{-1}\left(\frac{1}{\sqrt{r_\tau}}\right)\right)$$

where $\zeta_1(t_k + T) = 1$ and $\zeta_2(t_k + T) = -1$. If the penalty parameters satisfy $0 < (r_\tau - r_a)/(r_\tau r_a) < 1$, it is easy to verify that $\zeta_1(t) > 0$ for all $t \in [t_k, t_k + T]$; here, we used the fact that $\tanh(x)$ is defined for $|x| < 1$. Taking the time derivative, we have $\dot{V}_1 = \dot{d}_1' d_1 = -((r_\tau - r_a)/(r_\tau r_a))\zeta_1\, d_1' d_1 < 0$. This

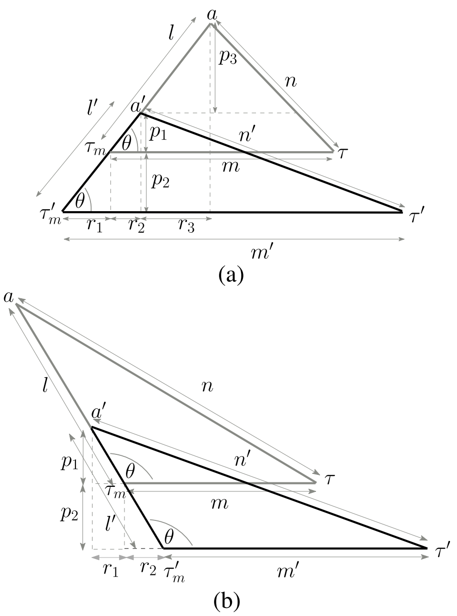


Fig. 2. Trajectories of players. Labels τ_m, τ and a are used to illustrate the position of minimum distance target, a target in $\mathscr{T}\backslash\tau_m$ and the attacker at time instant t_k. Labels τ'_m, τ' and a' illustrate the position of the same players at a time instant $t \in (t_k, t_k+1)$. (a) $0 < \theta < (\pi/2)$. (b) $(\pi/2) < \theta < \pi$.

implies that the distance between the attacker a and the target τ_m decreases strictly with time for the time duration $[t_k, t_{k+1})$. Next, when the planning horizon length $T > 0$ satisfies the condition $\sqrt{r_\tau}k\pi < T + \sqrt{r_\tau}\tan^{-1}(1/\sqrt{r_\tau}) < \sqrt{r_\tau}(k+1/2)\pi$ and $\sqrt{r_\tau}k\pi < T - \delta + \sqrt{r_\tau}\tan^{-1}(1/\sqrt{r_\tau}) < \sqrt{r_\tau}(k+1/2)\pi$ we have that $\zeta_2(t) < 0$ for all $t \in [t_k, t_{k+1}]$ with $t_{k+1} = t_k + \delta$. After rearranging these inequalities we obtain the condition (62). Taking the time derivative, we have $\dot{V}_2 = \dot{d}'_2 d_2 = -(1/r_\tau)\zeta_2\, d'_2 d_2 > 0$ for $t \in [t_k, t_{k+1})$. This implies that the distance between the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$ increases strictly with time for the time duration $[t_k, t_{k+1})$. Using the same approach as above it is easy to verify that when $T > 0$ satisfies (63) then the distance between the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$ decreases strictly with time for the time duration $[t_k, t_{k+1})$. ■

Assumption 2: The targets are symmetric, that is, $r_{\tau_i} = r_{\tau_j}$ for all $\tau_i, \tau_j \in \mathscr{T}$ and $i \neq j$. The penalty parameters of a target $\tau \in \mathscr{T}$ and the attacker a satisfy the condition $0 < (r_\tau - r_a)/(r_\tau r_a) < 1$ and the policy horizon length $T > 0$ satisfies (62).

As an immediate consequence of Lemma 1 and Assumption 2, we have the following result.

Theorem 5: Let Assumptions 1 and 2 hold true. Let t_k be the time instant when the game switches to the interception mode. Then, the target which was at minimum distance to the attacker a at the time instant t_k continues to remain so at the time instant t_{k+1}.

Proof: From Corollary 1, the angle between the lines joining the attacker a, the minimum distance target τ_m, and the target $\tau \in \mathscr{T}\backslash\tau_m$ remains constant throughout the time duration $[t_k, t_{t+1})$. Let us denote this angle by θ. First, we consider the case when $\theta \in (0, \pi/2)$ as illustrated in Fig. 2(a). Since τ_m is the minimum distance target we have $n > l$. From Theorem 3 and Theorem 4 we have $l > l'$ and $m' > m$. Next, from the triangle $\triangle c\tau_m a$, we have $n^2 - l^2 = (p_1+p_3)^2 + (m - (r_2+r_3))^2 - (p_1+p_3)^2 - (r_2+r_3)^2 = m^2 - 2m(r_2+r_3) = m(m - 2l\cos(\theta))$. As $n > l$ and $m > 0$, we have that $m > 2l\cos(\theta)$. Now, using the fact that $m' > m$ and $l > l'$, we get

$$m' > 2l'\cos(\theta). \tag{65}$$

From the triangle $\triangle a'\tau'_m\tau'$, we have $n'^2 - l'^2(p_1+p_2)^2 + (m' - (r_1+r_2))^2 - (p_1+p_2)^2 - (r_1+r_2)^2 = m'^2 - 2m'(r_1 + r_2) = m'(m' - 2l'\cos(\theta))$. From (65) this implies $n' > l'$. Next, we consider the case when $\theta \in (\pi/2, \pi)$ as illustrated in Fig. 2(b). From the triangles $\triangle a'\tau_m\tau$ and $\triangle a'\tau'_m\tau'$ we have $n'^2 - l'^2 = (p_1+p_2)^2 + (m+(r_1+r_2))^2 - (p_1+p_2)^2 - (r_1+r_2)^2 = m(m + 2(r_1+r_2)) > 0$.

Clearly, this implies $n' > l'$. When $\theta = \pi/2$, we have $r_1 + r_2 = 0$, then $n'^2 - l'^2 = m^2 > 0$. This implies that for $\theta \in (0, \pi)$ the statement of the theorem holds true. When $\theta = k\pi,\ k = 0,\ 1$, all the players lie on the same line and Lemma 1 provides the desired result. ■

Remark 12: Here, we emphasize that Fig. 2 connects the results obtained in Theorem 4, Corollary 1 and Lemma 1 as build-up toward the result provided in Theorem 5.

Remark 13: When the planning horizon length T is appropriately chosen as (62), Theorem 5 implies that the target which is at a minimum distance with the attacker, at time instant t_k, will remain so for all the time duration $t \in [t_k, t_k+1)$.

In the next result we show that switching policy defined by (24) renders the interception mode invariant, that is, once the game switches to the interception mode then further switchings cannot happen. Further, we also show that the attacker locks on to a target for the remaining duration of the game.

Theorem 6: Let Assumptions 1 and 2 hold true. Let t_k be the time instant when the game switches to the interception mode from the rescue mode according to the switching rule $\Psi(X(t))$ defined by (24). Let the minimum distance target at time t_k be

$$\tau^* := \arg\min_{\tau\in\mathscr{T}} ||X_a(t_k) - X_\tau(t_k)||_2. \tag{66}$$

Then, the interception mode is invariant. Further, the attacker locks on to the target τ^* for the remaining part of the game.

Proof: As the game enters the interception mode at t_k we have from (24) that $||X_a(t_k) - X_{\tau^*}(t_k)||_2 \leq \kappa\sigma_a$. Next, for the time duration $[t_k, t_{k+1})$ we know from Lemma 1 that $||X_a(t) - X_{\tau^*}(t)||_2$ is a strictly decreasing function of time for $t \in [t_k, t_{k+1})$. Moreover, from Theorem 5 we have that the target τ^* remains to be the minimum distance target at t_{k+1} as well. In particular, we have that

$$||X_a(t_{k+1}) - X_{\tau^*}(t_{k+1})||_2 < ||X_a(t_k) - X_{\tau^*}(t_k)||_2 \leq \kappa\sigma_a$$

implying that mode switching cannot happen at the time instant t_{k+1} and the game continues in the interception mode during the time period $[t_{k+1}, t_{k+2})$. Using the same arguments at next time instant t_{k+2} we infer that interception mode is invariant. Furthermore, as the target τ^* given by (66) remains to be the minimum distance target at every time instant, the attacker locks on to τ^* from t_k onwards. ■

So far, we have analyzed the situation where the game enters the interception mode at time instant t_k. In the following theorem, we study the nature of trajectories when the game enters rescue mode at t_k.

Theorem 7: Let Assumption 1 hold, and let $r_{\tau_i} = r_{\tau_j}$ for all $\tau_i, \tau_j \in \mathscr{T}$ and $i \neq j$. Let t_k be the time instant when the game enters the rescue mode. Then, the target τ_m which was at minimum distance to the attacker a at time instant t_k remains at a constant distance and orientation with other targets $\tau \in \mathscr{T}\backslash\tau_m$ for the time duration $[t_k, t_{k+1})$.

Proof: We consider the interaction between the players $\{\tau_m, \tau, d, a\}$. The Riccati differential equation (14) associated with player $i \in \{\tau_m, \tau, d, a\}$ is given by

$$\dot{P}_i = -\widetilde{Q}_i + P_i\left(S_{\tau_m}P_{\tau_m} + S_\tau P_\tau + S_d P_d + S_a P_a\right) \quad (67)$$

where $P_i(t_k + T) = \widetilde{Q}_{iT}$. In rescue mode, we have

$$\widetilde{Q}_{\tau_m} = \widetilde{Q}_{\tau_m T} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 \end{bmatrix}$$

$$\widetilde{Q}_{\tau} = \widetilde{Q}_{\tau T} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & -\mathbf{I}_2 \end{bmatrix}$$

$$\widetilde{Q}_{d} = \widetilde{Q}_{dT} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & -\mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & -\mathbf{I}_2 & \mathbf{0}_2 \\ -\mathbf{I}_2 & -\mathbf{I}_2 & 2\mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix} \text{ and}$$

$$\widetilde{Q}_{a} = \widetilde{Q}_{aT} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix}.$$

Using the open-loop Nash equilibrium controls, the state vector is written as

$$\dot{X}(t) = -\left(S_{\tau_m}P_{\tau_m} + S_\tau P_\tau + S_d P_d + S_a P_a\right)X(t).$$

We partition the matrix $P_i(t)$ for $i \in \{\tau_m, \tau, d, a\}$ similar to (27) to obtain

$$\begin{bmatrix} \dot{X}_{\tau_m} \\ \dot{X}_\tau \\ \dot{X}_d \\ \dot{X}_a \end{bmatrix} = -\begin{bmatrix} r_{\tau_m}^{-1}P_{\tau_m}^{11} & r_{\tau_m}^{-1}P_{\tau_m}^{12} & r_{\tau_m}^{-1}P_{\tau_m}^{13} & r_{\tau_m}^{-1}P_{\tau_m}^{14} \\ r_{\tau}^{-1}P_{\tau}^{21} & r_{\tau}^{-1}P_{\tau}^{22} & r_{\tau}^{-1}P_{\tau}^{23} & r_{\tau}^{-1}P_{\tau}^{24} \\ r_{d}^{-1}P_{d}^{31} & r_{d}^{-1}P_{d}^{32} & r_{d}^{-1}P_{d}^{33} & r_{d}^{-1}P_{d}^{34} \\ r_{a}^{-1}P_{a}^{41} & r_{a}^{-1}P_{a}^{42} & r_{a}^{-1}P_{a}^{43} & r_{a}^{-1}P_{a}^{44} \end{bmatrix}\begin{bmatrix} X_{\tau_m} \\ X_\tau \\ X_d \\ X_a \end{bmatrix}.$$

Using the above, we can write

$$\begin{aligned} &\dot{X}_{\tau_m} - \dot{X}_\tau \\ &= -\big[\big(r_{\tau_m}^{-1}P_{\tau_m}^{11} - r_\tau^{-1}P_\tau^{21}\big)X_{\tau_m} + \big(r_{\tau_m}^{-1}P_{\tau_m}^{12} - r_\tau^{-1}P_\tau^{22}\big)X_\tau \\ &\quad + \big(r_{\tau_m}^{-1}P_{\tau_m}^{13} - r_\tau^{-1}P_\tau^{23}\big)X_d + \big(r_{\tau_m}^{-1}P_{\tau_m}^{14} - r_\tau^{-1}P_\tau^{24}\big)X_a\big]. \end{aligned}$$

Since $r_{\tau_m} = r_\tau$ we have

$$\begin{aligned} \dot{X}_{\tau_m} - \dot{X}_\tau = &-r_{\tau_m}^{-1}\big[\big(P_{\tau_m}^{11} - P_\tau^{21}\big)X_{\tau_m} + \big(P_{\tau_m}^{12} - P_\tau^{22}\big)X_\tau \\ &+ \big(P_{\tau_m}^{13} - P_\tau^{23}\big)X_d + \big(P_{\tau_m}^{14} - P_\tau^{24}\big)X_a\big]. \end{aligned} \quad (68)$$

Denoting $\Gamma_2(t) = (S_{\tau_m}P_{\tau_m} + S_\tau P_\tau + S_d P_d + S_a P_a)$ we write (67) for P_{τ_m} and P_τ as

$$\dot{P}_{\tau_m} = -\widetilde{Q}_{\tau_m} + P_{\tau_m}\Gamma_2(t) \quad (69)$$

$$\dot{P}_{\tau} = -\widetilde{Q}_{\tau} + P_{\tau}\Gamma_2(t). \quad (70)$$

Again using the partitioning (27) and premultiplying the (69) with the matrix $[\mathbf{I}_2 \ \ \mathbf{0}_2 \ \ \mathbf{0}_2 \ \ \mathbf{0}_2]$ and premultiplying (70) with $[\mathbf{0}_2 \ \ \mathbf{I}_2 \ \ \mathbf{0}_2 \ \ \mathbf{0}_2]$ we obtain

$$\begin{aligned} &\big[\,\dot{P}_{\tau_m}^{11} \ \ \dot{P}_{\tau_m}^{12} \ \ \dot{P}_{\tau_m}^{13} \ \ \dot{P}_{\tau_m}^{14}\,\big] \\ &= \big[\,\mathbf{0}_2 \ \ \mathbf{0}_2 \ \ -\mathbf{I}_2 \ \ \mathbf{I}_2\,\big] + \big[\,P_{\tau_m}^{11} \ \ P_{\tau_m}^{12} \ \ P_{\tau_m}^{13} \ \ P_{\tau_m}^{14}\,\big]\Gamma_2(t) \\ &\big[\,\dot{P}_{\tau}^{21} \ \ \dot{P}_{\tau}^{22} \ \ \dot{P}_{\tau}^{23} \ \ \dot{P}_{\tau}^{24}\,\big] \\ &= \big[\,\mathbf{0}_2 \ \ \mathbf{0}_2 \ \ -\mathbf{I}_2 \ \ \mathbf{I}_2\,\big] + \big[\,P_{\tau}^{21} \ \ P_{\tau}^{22} \ \ P_{\tau}^{23} \ \ P_{\tau}^{24}\,\big]\Gamma_2(t). \end{aligned}$$

Taking the difference of the above two differential equations, we obtain

$$\begin{aligned} &\big[\,\dot{P}_{\tau_m}^{11} - \dot{P}_\tau^{21} \ \ \ \dot{P}_{\tau_m}^{12} - \dot{P}_\tau^{22} \ \ \ \dot{P}_{\tau_m}^{13} - \dot{P}_\tau^{23} \ \ \ \dot{P}_{\tau_m}^{14} - \dot{P}_\tau^{24}\,\big] \\ &= \big[\,P_{\tau_m}^{11} - P_\tau^{21} \ \ \ P_{\tau_m}^{12} - P_\tau^{22} \ \ \ P_{\tau_m}^{13} - P_\tau^{23} \ \ \ P_{\tau_m}^{14} - P_\tau^{24}\,\big]\Gamma_2(t) \end{aligned}$$

with terminal conditions $P_{\tau_m}^{1j}(t_k + T) - P_\tau^{2j}(t_k + T) = \mathbf{0}_2$ for $j = 1, 2, 3, 4$. This implies that $P_{\tau_m}^{1j}(t) - P_\tau^{2j}(t) = \mathbf{0}_2$ for all $t \in [t_k, t_k + T]$. Using this in (68) we obtain $\dot{X}_{\tau_m} - \dot{X}_\tau = \mathbf{0}_{2\times1}$. This implies that the target τ_m remains constant distance and orientation with target τ in the rescue mode. ■

Remark 14: In the rescue mode, all the targets maximize their distance with the attacker, and minimize their distance with the defender. So, these two opposing behaviors result in the distance between the targets τ_m and $\tau \in \mathscr{T}\backslash\{\tau_m\}$ to remain constant. We let weights in the target's objectives (5) as $Q_{\tau d} = Q_{\tau dT} = q_{\tau d}\mathbf{I}_2$ and $Q_{\tau a} = Q_{\tau aT} = q_{\tau c}\mathbf{I}_2$, with $q_{\tau d} > 0$ and $q_{\tau a} > 0$. If $q_{\tau d} > q_{\tau a}$ then targets give more weightage on rendezvousing with the defender than evading the attacker, and vice-versa when the weights satisfy $q_{\tau d} < q_{\tau a}$.

VI. Simulation Results

In this section, we illustrate the performance of switching strategies, developed in Section IV, through numerical experiments. We consider a six-player game consisting of four targets, one defender and one attacker. We analyze two scenarios. In Scenario-1, we verify the results developed in Section V and analyze the effect of varying the parameters $Q_{\tau d} = Q_{\tau dT} = q_{\tau d}\mathbf{I}_2$, $Q_{\tau a} = Q_{\tau aT} = q_{\tau a}\mathbf{I}_2$ for $\tau \in \mathscr{T}$ and the planning horizon T. In Scenario-2, we analyze the effect of switching function parameter κ in (24), the degree of alertness of the defender, on the outcome of the game.

Scenario-1: Initially, the four targets τ_1, τ_2, τ_3, and τ_4 are located at $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, respectively. The defender d and the attacker a are located at $(-4, 2)$ and $(4, 4)$, respectively. The parameter values for the baseline case are taken as follows: $q_{\tau d} = 1$, $q_{\tau a} = 1$, $R_\tau = 400\mathbf{I}_2$, $Q_{d\tau} = Q_{d\tau T} = Q_{a\tau} = Q_{a\tau T} = \mathbf{I}_2$ for $\tau \in \{\tau_1, \tau_2, \tau_3, \tau_4\}$, $R_d = 300\mathbf{I}_2$, $R_a = 200\mathbf{I}_2$, $Q_{da} = Q_{daT} = \mathbf{I}_2$, $T = 5$, $\delta = 0.02$, $\sigma_d = \sigma_a = 0.5$ and $\kappa = 5$. For the baseline case, Fig. 3(a) illustrates the trajectories of the players, and Fig. 3(b) illustrates the distances between the players. The defender starts in the rescue mode and switches to interception mode

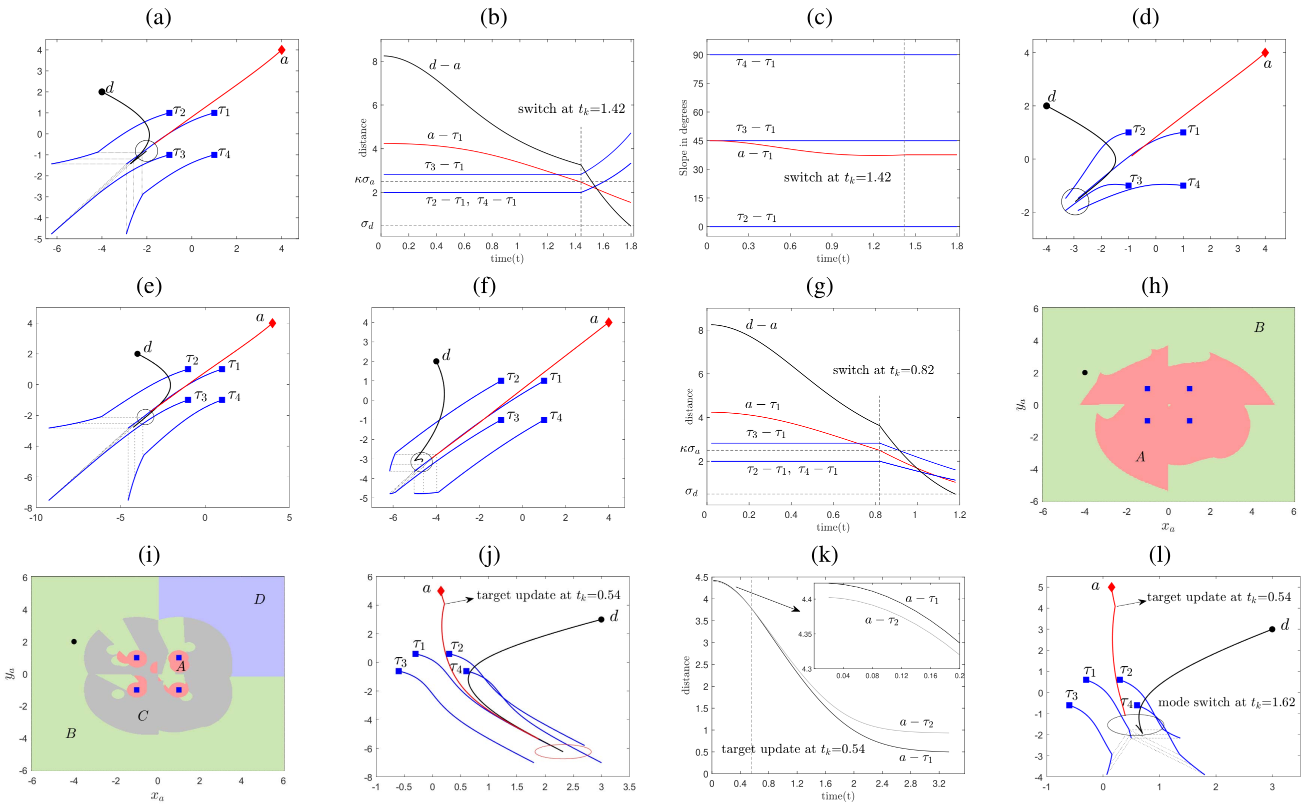


Fig. 3. In (a), (e), (f), and (l) dotted lines illustrate the lines joining the minimum distance target and the other targets in the interception mode. Black and red circles represent the capture zone of the defender and attacker, respectively. In (h) and (i) colored regions indicate different game outcomes based attacker's initial location. Attacker's initial location, 1) in region A, implies the outcome: capture of the target by the attacker; 2) in region B, implies the outcome: interception of the attacker by the defender; 3) in region C, implies no outcome; and 4) in region D, implies the outcome: rescue of all the targets by the defender. (a) Scn. 1: trajectories (baseline). (b) Scn. 1: distances (baseline). (c) Scn. 1: slopes (baseline). (d) Scn. 1: with $q_{\tau d} = 4$. (e) Scn. 1: with $q_{\tau a} =$ 1:5. (f) Scn. 1: trajectories with $T = 50$. (g) Scn. 1: distances with $T = 50$. (h) Scn. 1: GoK (baseline). (i) Scn. 1: GoK ($q_{\tau d} =$ 2:25; $q_{\tau a} = 2$). (j) Scn. 2: $k = 1$. (k) Scn. 2: $k = 1$ target update. (l) Scn. 2: $k = 3$.

at $t_k = 1.42$, when the distance between the attacker a and its minimum distance target τ_1 is less than or equal to $\kappa\sigma_a = 2.5$. The distances between τ_1 and other targets τ_2, τ_3, and τ_4 remain constant in the rescue mode verifying Theorem 7. From Fig. 3(c), the slope of the lines joining the attacker and the target τ_1 is constant at 37.5878° in the interception mode. This verifies Theorem 3. Again, the slopes of the lines joining the target τ_1 with τ_2, τ_3, and τ_4 remain constant at 0°, 45°, and 90°, respectively, verifying Theorem 4. Next, the planning horizon T satisfies the condition (62) with $k = 0$, as $T = 5 \in (-0.9792, 30.4368)$. This implies, Assumption 2 holds true. From Fig. 3(b), in the interception mode (after $t > 1.42$), the distance between the attacker and the target τ_1 decreases with time. Further, the distance between the targets τ_1 with τ_2, τ_3, and τ_4 increases with time. These observations verify Lemma 1 and Theorem 5. Further, the attacker locks on to the target τ_1 after $t_k > 1.42$, thus verifying Theorem 6. From Fig. 3(b) the distance between the defender and the attacker equals the capture radius σ_d at time $t_k = 1.78$, implying that the defender intercepts the attacker. From Remark 14, when $q_{\tau d} > q_{\tau a} = 1$, the inter target distance decreases as the targets emphasize rendezvousing with the defender more than evading the attacker. Fig. 3(d) illustrates this observation when $q_{\tau d}$ is taken as 2.75, where the outcome of the game results in rescue of all the targets. When the parameter $q_{\tau a}$ is set to $1.2 > q_{\tau d} = 1$, then the inter target distance increases as the targets now emphasize evading the attacker more than rendezvousing with the defender. Fig. 3(e) illustrates this observation where the defender intercepts the attacker. Next, we analyze the effect of varying the planning horizon length T. In the baseline case, T satisfies the condition (62). Now, we set $T = 50$ so as to satisfy the other condition (63) with $k = 1$, that is, $T = 50 \in (30.4368, 61.8527)$. Fig. 3(f) and (g) illustrates the trajectories of the players and distances between the players, respectively. From Fig. 3(g), it can be seen that the distance between the target τ_1 with τ_2 and τ_3 decreases with time in the interception mode (after $t_k > 0.82$). This observation again verifies Lemma 1.

Recall from Remark 2 that a GoK can be embedded within the framework of a GoD. To illustrate this, we vary the initial location of the attacker in the region $(x_a, y_a) \in [-6, 6] \times [-6, 6]$, while keeping all other parameter values fixed at the baseline values, and identify the game outcome using the switching strategies. Notice, we chose to vary only the initial location of the attacker as the state space is 12-D. In Fig. 3(h) the regions A and B indicate initial locations of the attacker which result in the outcomes capture of the target by the attacker, and interception of the attacker by the defender, respectively. Next, we change the parameters to $q_{\tau d} = 2.25$, $q_{\tau a} = 2$. In Fig. 3(i) the regions C and D indicate the initial locations of the attacker which result in the no outcome, and rescue of all targets by the defender, respectively. When the attacker starts in the region D, it is located relatively far away from the targets. From Remark 14, as $q_{\tau a} = 2.25 > 1$, the targets emphasize on rendezvousing with the defender in the rescue mode. Hence, for all the initial locations of the defender in the region D game outcome is a rescue of all the targets. Other regions can be analyzed in a similar fashion.

Scenario-2: Initially, the four targets τ_1, τ_2, τ_3, and τ_4 are located at $(-0.3, 0.6)$, $(0.3, 0.6)$, $(-0.6, -0.6)$, and $(0.6, -0.6)$, respectively. The defender d and the attacker a are located at $(3, 3), (0.15, 5)$, respectively. The remaining parameters are set according to the baseline case in Scenario-1. First we set the parameter $\kappa = 1$ and the game starts in rescue mode. Fig. 3(j) illustrates the trajectories of the players. Fig. 3(k) illustrates the distance between the attacker a and the targets τ_1 and τ_2. At the time instant $t_k = 0.54$, the attacker updates its minimum distance target from τ_2 to τ_1. The game terminates at $t_k = 3.34$ with attacker capturing the target τ_1. Next, when the parameter κ is increased to 3, indicating a highly alert defender, it can be observed from Fig. 3(l) that the defender switches from rescue mode to interception mode at $t_k = 1.62$ and eventually intercepts the attacker at $t = 1.86$. As the parameter κ only influences the defender's ability to switch the operational behavior, the behavior of the player before the mode switch at time instant $t_k = 1.62$ is identical to the situation where $\kappa = 1$. This implies that the attacker updates the minimum distance target from τ_2 to τ_1 at time instant $t_k = 0.54$ for this case as well. It can be observed that in the interception mode the lines joining the minimum distance target τ_1 with the other targets remain parallel verifying Theorem 4. Further, the inter target distance remains constant in the rescue mode verifying Theorem 7.

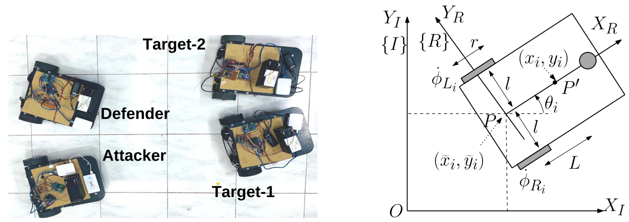
VII. Experimental Study

In this section, we illustrate the dynamic game model and the implementation of the Algorithm 1 through experiments with players taken as DDMRs. We present the robot model, discuss the experimental setup and illustrate some of the results obtained in Section V.

A. Robot Model and Feedback Linearization

A DDMR with two motorized fixed standard wheels and one unpowered omni-directional castor wheel is shown in Fig. 4(b). Here, $\{I\}$ denotes the inertial frame of reference with origin O and basis (X_I, Y_I). $\{R\}$ corresponds to the local frame of reference having position P and basis (X_R, Y_R). The position of the robot in the inertial frame of reference is given by $(\tilde{x}_i, \tilde{y}_i)$ while θ_i, $i = \{\mathscr{T}, d, a\}$ corresponds to the angular difference between frames. The dynamics of robot i is given by; see [38], [39]

$$\dot{\tilde{x}}_i = \left(\frac{r\dot{\phi}_{Ri} + r\dot{\phi}_{Li}}{2}\right)\cos\theta_i, \quad \dot{\tilde{y}}_i = \left(\frac{r\dot{\phi}_{Ri} + r\dot{\phi}_{Li}}{2}\right)\sin\theta_i \tag{71a}$$



(a) (b)

Fig. 4. (a) Illustrates the 4 DDMRs used in the experiments. (b) Illustrates the DDMR model showing inertial frame of reference with axes (X_I, Y_I) and robot's frame of reference with axes (X_R, Y_R).

$$\dot{\theta}_i = \frac{r\dot{\phi}_{Ri} - r\dot{\phi}_{Li}}{2l} \tag{71b}$$

where $2l$ is the distance between the wheels and r is the diameter of the wheel. The angular velocities of the right wheel $(\dot{\phi}_{Ri})$ and left wheel $(\dot{\phi}_{Li})$ are the control inputs with $(\tilde{x}_i, \tilde{y}_i, \theta_i)$ as the pose of the robot in robot frame $\{R\}$ at time t. Let v_i and ω_i be the translational and angular velocities of the robot, respectively. Then we have $v_i = (r\dot{\phi}_{Ri} + r\dot{\phi}_{Li})/2$, $\omega_i = (r\dot{\phi}_{Ri} - r\dot{\phi}_{Li})/(2l)$. The DDMR dynamics can be rewritten as the following unicycle dynamics [40, Ch. 2]:

$$\dot{x}_i = v_i \cos\theta_i, \quad \dot{y}_i = v_i \sin\theta_i, \quad \dot{\theta}_i = \omega_i. \tag{72}$$

However, for implementation purposes, the actual control inputs $\dot{\phi}_{Ri}$, $\dot{\phi}_{Li}$ are obtained from (71) and (72) as

$$\dot{\phi}_{Ri} = \frac{1}{r}(v_i + l\omega_i), \quad \dot{\phi}_{Li} = \frac{1}{r}(v_i - l\omega_i). \tag{73}$$

The robot dynamics given by (71) is nonlinear, and a dynamic game formulation is difficult to solve in general. We, therefore, use feedback linearization [40, Ch. 2] and then apply our LQDG framework.

Let P be the origin of the robot in robot frame and P' be the center of mass at a distance L from the origin P as shown in Fig. 4(b). For robot $i \in \mathscr{P}$, the coordinates of P' are $x_i = \tilde{x}_i + L\cos\theta_i$, $y_i = \tilde{y}_i + L\sin\theta_i$. Upon differentiating these equations and using (71), (72) and (73) we get

$$\dot{x}_i = v_i \cos\theta_i - L\omega_i \sin\theta_i, \quad \dot{y}_i = v_i \sin\theta_i + L\omega_i \cos\theta_i. \tag{74}$$

We define the following state feedback laws:

$$v_i = \cos(\theta_i)u_{1i} + \sin(\theta_i)u_{2i} \tag{75a}$$

$$\omega_i = \frac{1}{L}(-\sin(\theta_i)u_{1i} + \cos(\theta_i)u_{2i}) \tag{75b}$$

and then using (75a) in (74) we get

$$\dot{x}_i = u_{1i}, \quad \dot{y}_i = u_{2i}. \tag{76}$$

The LQDG formulation considers the point P' and provides the Nash equilibrium controls (u_{1i}, u_{2i}) for robot i. For implementation, the actual controls $(\dot{\phi}_{Ri}, \dot{\phi}_{Li})$ are obtained using (75a) and (75b) in (73) as

$$\dot{\phi}_{Ri} = \frac{\cos(\theta_i)}{r}\left(u_{1i} + \frac{u_{2i}l}{L}\right) + \frac{\sin(\theta_i)}{r}\left(u_{2i} - \frac{u_{1i}l}{L}\right) \tag{77a}$$

$$\dot{\phi}_{Li} = \frac{\cos(\theta_i)}{r}\left(u_{1i} - \frac{u_{2i}l}{L}\right) + \frac{\sin(\theta_i)}{r}\left(u_{2i} + \frac{u_{1i}l}{L}\right). \tag{77b}$$

From (77a) and (77b), it is evident that the feedback linearization parameter L must be chosen carefully.

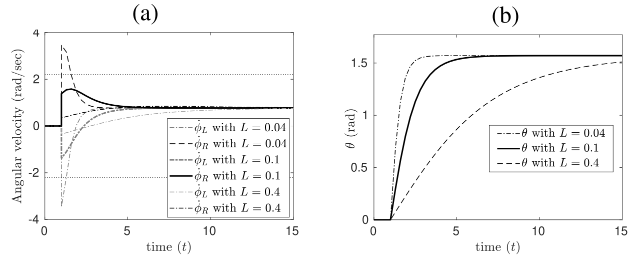


Fig. 5. DDMR controls and orientation. (a) Angular velocities. (b) Orientation.

B. Experimental Setup and Implementation Details

The experiments employ four DDMRs that serve as an attacker, a defender, and two targets; see Fig. 4(a). The distance between the wheels of a robot is $2l = 0.36$ m and the diameter of each wheel is $r = 0.13$ m. Each of the robots has access to their initial location information. As the game progresses, each robot tracks and determines its local position and orientation with the help of Autonics E40H12 rotary encoders mounted on its two wheels. This state information is made available to the remaining players (and vice versa) over a wireless communication network. The implementation details of Algorithm 1 are given as follows. At a given time instant t_k, the computation of open-loop Nash equilibrium strategies over the planning horizon $[t_k, t_k + T]$ are performed using a Raspberry Pi 3 B+ board installed on each robot. Next, an on-board Arduino UNO is employed to enforce the control inputs on physical robots for the duration $[t_k, t_{k+1})$. Finally, a coordination protocol is adopted to synchronize the time instants t_k for the execution of control inputs (77) on each robot. The feedback linearization parameter L relates the control inputs obtained from Algorithm 1 to the respective wheel velocities of the robots in the experimental setup. While a small value of L is desirable, it results in higher wheel velocities. Fig. 5(a) illustrates the wheel angular velocities for three values of L for a step input $u_1 = 0$ and $u_2 = 0.1$ m/s at 1 s. Here, these choices of inputs result in the maximum possible/worst case rotation of the robot, which is 90°. The dotted horizontal lines indicate the maximum achievable angular velocity of 21 rpm(=2.2 rad/s) for the dc motors mounted on the robot wheels. When $L = 0.04$ m, the maximum wheel angular velocities shoot up to a maximum of 3.46 rad/s which are impractical during implementation. Fig. 5(b) illustrates the time taken by the robot to reach the desired rotation of 90°. It can be observed that higher values of L the robot take a longer time to reach the desired orientation. We thus adopt an intermediate value of $L = 0.1$ m that requires a maximum speed of 1.58 rad/s (see Fig. 5(a)) which is well within the achievable speed of the dc motors (2.2 rad/s). Finally, the moving horizon time instant duration δ is taken as 0.5 s to accommodate the time spent in inter robot communication and enforcement of determined wheel velocities on physical robots.

C. Experiment-1 (With One Target)

In this experiment, we consider one target to illustrate the implementation of Algorithm 1. The initial P (P')-coordinates of the target, defender and the attacker are given by (0.4, 3, 0°) (0.5, 3, 0°), (3.4, 3, 180°) (3.3, 3, 180°), and (1.8, 4.3, 270°) (1.8, 4.2, 270°), respectively. The third coordinate indicates the orientation of the robots with respect to positive x-axis. The parameters are set as $\delta = 0.5$ s, $T = 45$ s, $R_\tau = 380\mathbf{I}_2$, $R_d = 350\mathbf{I}_2$, $R_a = 300\mathbf{I}_2$, $Q_{\tau dT} = Q_{\tau aT} = Q_{\tau d} = Q_{\tau a} = Q_{d\tau T} = Q_{d\tau} = Q_{a\tau T} = Q_{a\tau} = \mathbf{I}_2$ and $Q_{da} = Q_{daT} = 5\mathbf{I}_2$. The capture radii of the defender and attacker are set as $\sigma_d = \sigma_a = 0.5$ m with $\kappa = 3.2$. Since there is only one target the attacker always pursues this target. As the initial distance between the attacker and target is greater than $\kappa\sigma_a$ the game starts in the rescue mode. This distance equals $\kappa\sigma_a$ at 8 s and the game switches to interception mode; see Fig. 6(b). The game terminates with the

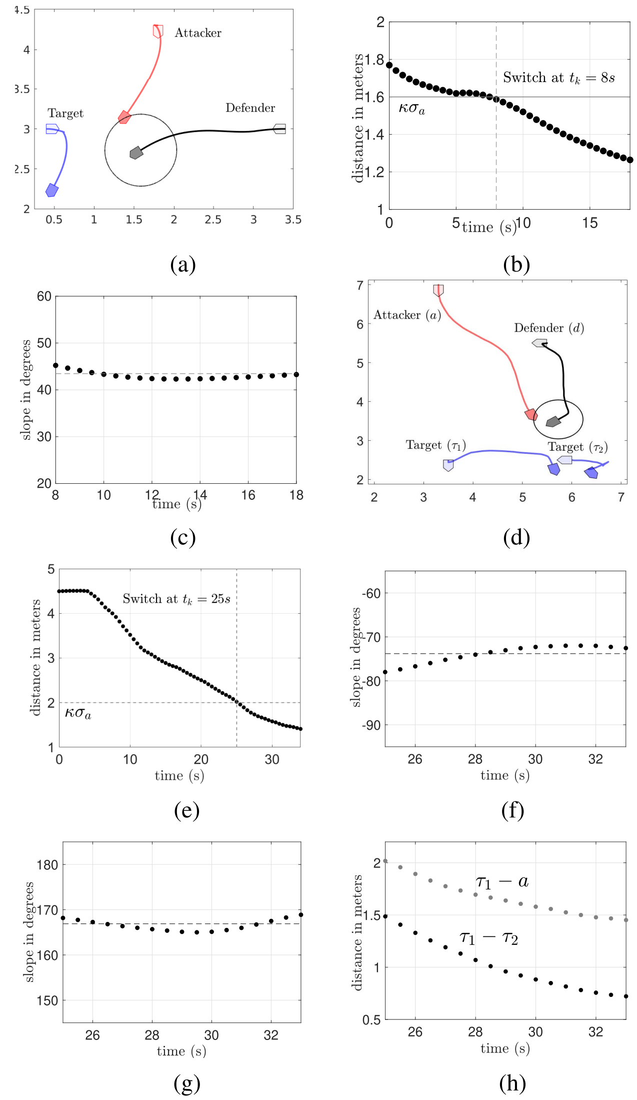


Fig. 6. *Experiment-1*: (a) illustrates the P-trajectories of the attacker, defender and the target. (b) Depicts the distance between the target and the attacker. (c) Depicts the slope of the line joining the target and the attacker in the interception mode. *Experiment-2*: (d) Illustrates the P-trajectories of the attacker, defender and the targets. (e) Illustrates the distance between the minimum distance target (τ_1) and the attacker. (f) Illustrates the slope of the line joining the target (τ_1) and the attacker in the interception mode. (g) Illustrates the slope of the line joining the targets (τ_1 and τ_2) in the interception mode. (h) Illustrates the distances between target τ_1 with the attacker a and the target τ_2.

attacker intercepted by the defender at time instant $t = 18$ s, that is, when the attacker lies within the capture radius of the defender; see Fig. 6(a). Fig. 6(c) illustrates the slope of the line joining the attacker and the target for the duration [8 s, 18 s] in the interception mode. We observe that the mean slope of this line is 43.4610° with standard deviation 0.8519°, implying that the slope is almost constant thus verifying Theorem 3. Video recording of the experiment is available at the link https://youtu.be/JX2O4fYb4g4.

D. Experiment-2 (With Two Targets)

In this experiment we consider two targets. The initial $P(P')$-coordinates of the targets (labeled as τ_1 and τ_2), defender and attacker are taken as (3.5, 2.5, 270°) (3.5, 2.4, 270°), (6, 2.5, 180°) (5.9, 2.5, 180°), (5.5, 5.5, 180°) (5.4, 5.5, 180°), and (3.3, 7, 270°) (3.3, 6.9, 270°), respectively. The initialization parameters are set as $\delta = 0.5$ s, $T = 45$ s, $R_{\tau_1} = R_{\tau_2} = 480\mathbf{I}_2$, $R_d = 350\mathbf{I}_2$, $R_a = 280\mathbf{I}_2$, $Q_{\tau_1 d} = Q_{\tau_2 d} = Q_{\tau_1 dT} = Q_{\tau_2 dT} = 2\mathbf{I}_2$, $Q_{da} = Q_{daT} = 3\mathbf{I}_2$ with other matrices taken as $\mathbf{I}_2$. The capture radii of the defender and the attacker are taken as $\sigma_d = \sigma_a = 0.5$ with the switching parameter set as $\kappa = 4$. Based on switching condition, initially, the defender attempts to rescue both the target robots while the attacker pursues its closest target τ_1. In our experiment, the target τ_1 remains to be the minimum distance target for the attacker throughout the game. The game continues in the rescue mode for 25 s when target τ_1 is at a distance of $\kappa\sigma_a = 2$ meters from the attacker; see Fig. 6(e). During the interval [25 s, 33 s], that is, during the interception mode, Fig. 6(f) illustrates the slope of the line joining between τ_1 and a. We observe that the mean value of the slope is −75.2123° with standard deviation 1.2968°, implying that the slope is almost constant thus verifying Theorem 3. Fig. 6(g) illustrates the slope of the line joining the targets τ_1 and τ_2. We observe that the mean value of the slope is 166.8873° with standard deviation 1.5289°, implying that the slope is almost constant verifying Theorem 4. We notice that the parameters satisfy $(r_\tau - r_a)/(r_\tau r_a) = 0.015 \in (0, 1)$, and the planning horizon T satisfies condition (63) with $k = 1$, as $T = 45 \in (33.9151, 67.8295)$. Fig. 6(h) illustrates the distance between the target τ_1 and the attacker decreases with time and the inter target distance also decreases with time. This observation verifies Lemma 1. Video recording of the experiment is available at the link https://youtu.be/MKmOo5ssQMY.

VIII. Conclusion

In this article, we have analyzed a multiple ATAD differential game where the defender adaptively switches operating in rescue and interception modes, and the attacker pursues the closest target during the course of the game. We model the interactions within each mode as LQDG and derive open-loop Nash equilibrium strategies of the players. Then, to enable switching we use the receding horizon approach to obtain switching strategies for the players. Under few assumptions on the problem parameters, we characterized the geometrical properties of the trajectories of the players. Further, we also derived conditions under which the attacker locks on to a target. We illustrated our results with numerical simulations. Further, we demonstrated the performance of switching strategies using DDMRs and verified our results.

The ATAD model studied in our article can be easily adapted to incorporate multiple defenders and attackers. For future work, we plan to investigate different cooperation situations between the targets and the defender, interactions where the attacker is also concerned about evading the defender, various criteria for switching and terminating the game, and the presence of obstacles.

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